Multi-wavelength emission from Jets and Magnetically Arrested disks of Cygnus X-1

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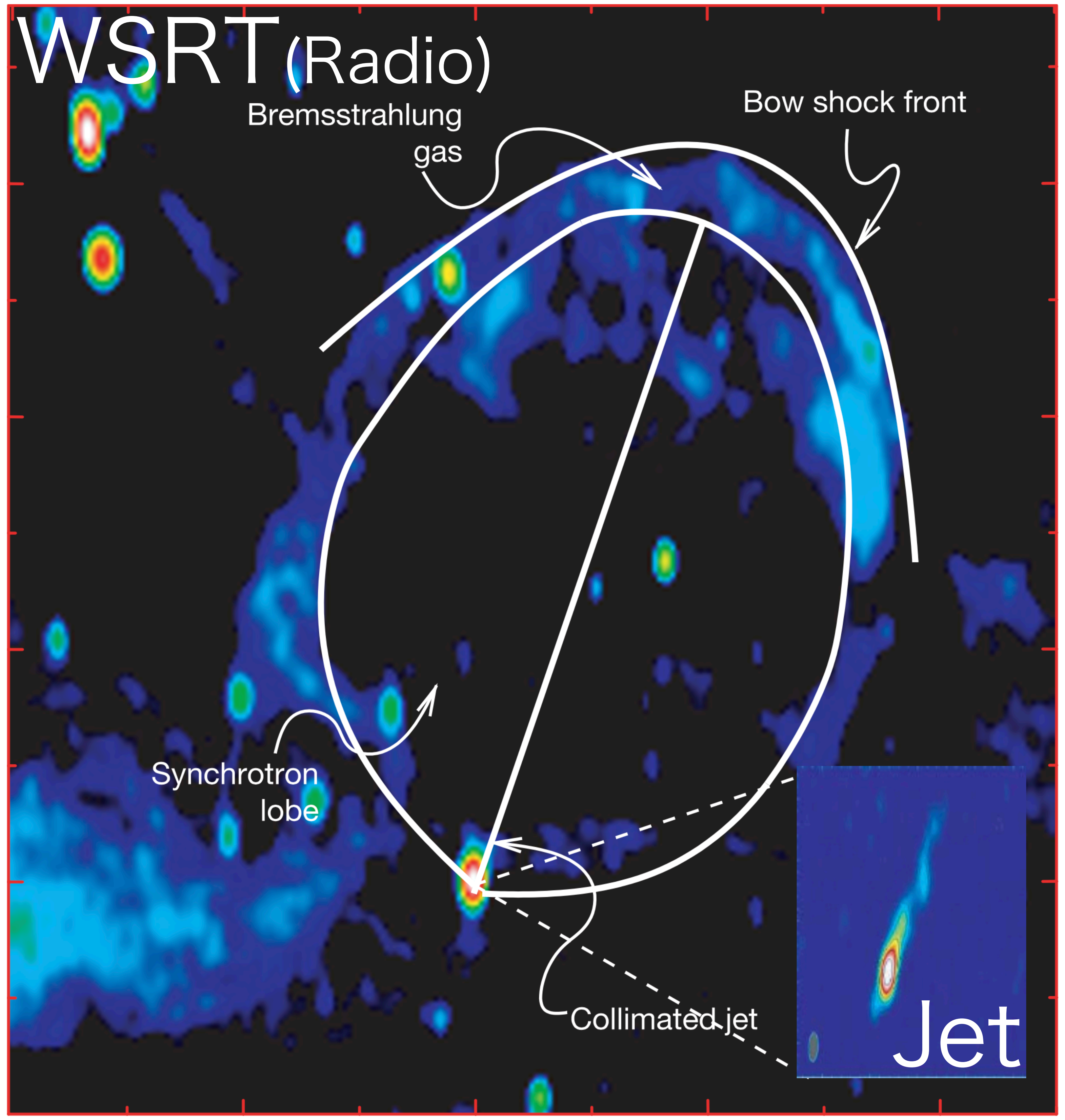
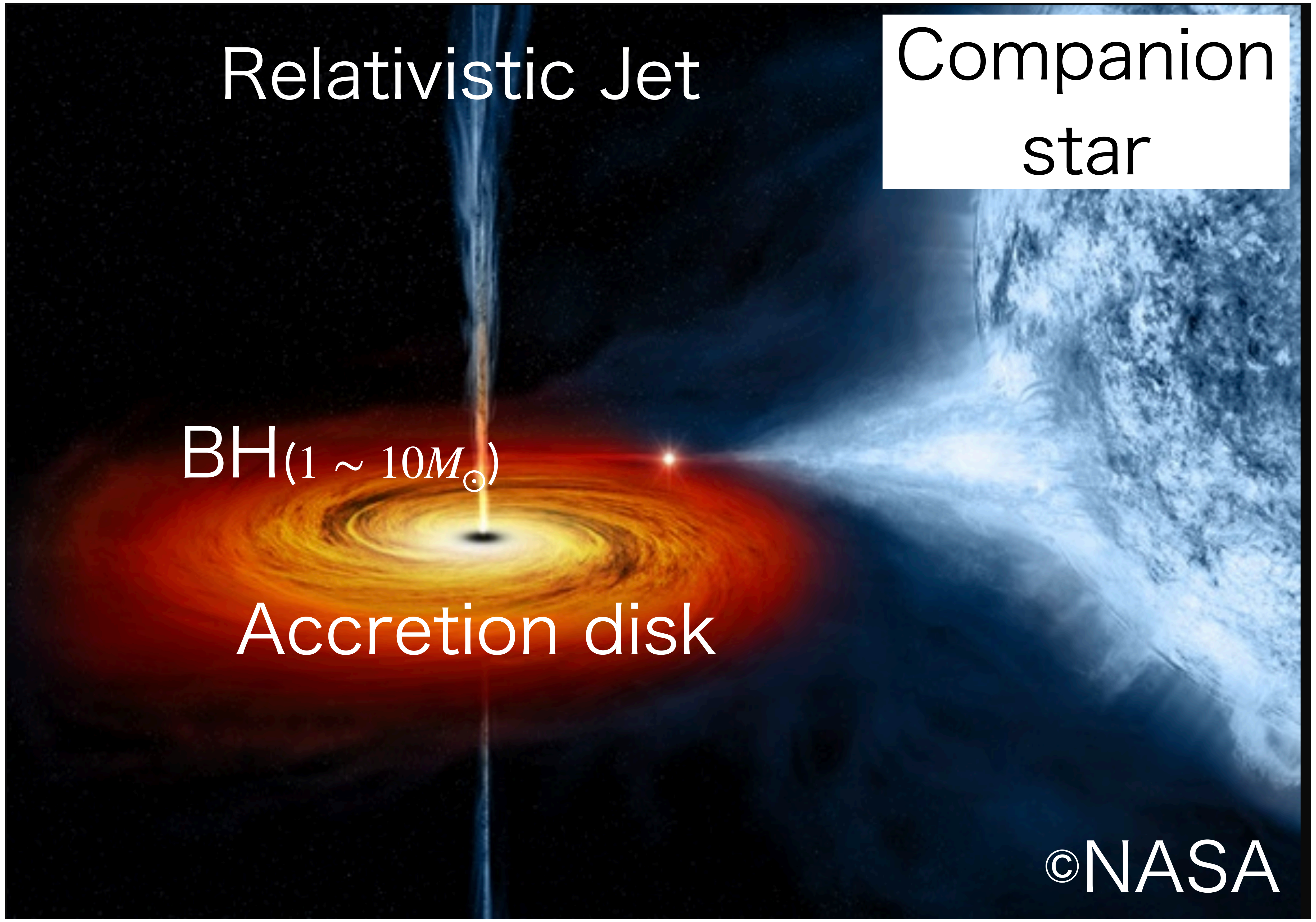
Black hole X-ray Binaries (BHXBs)



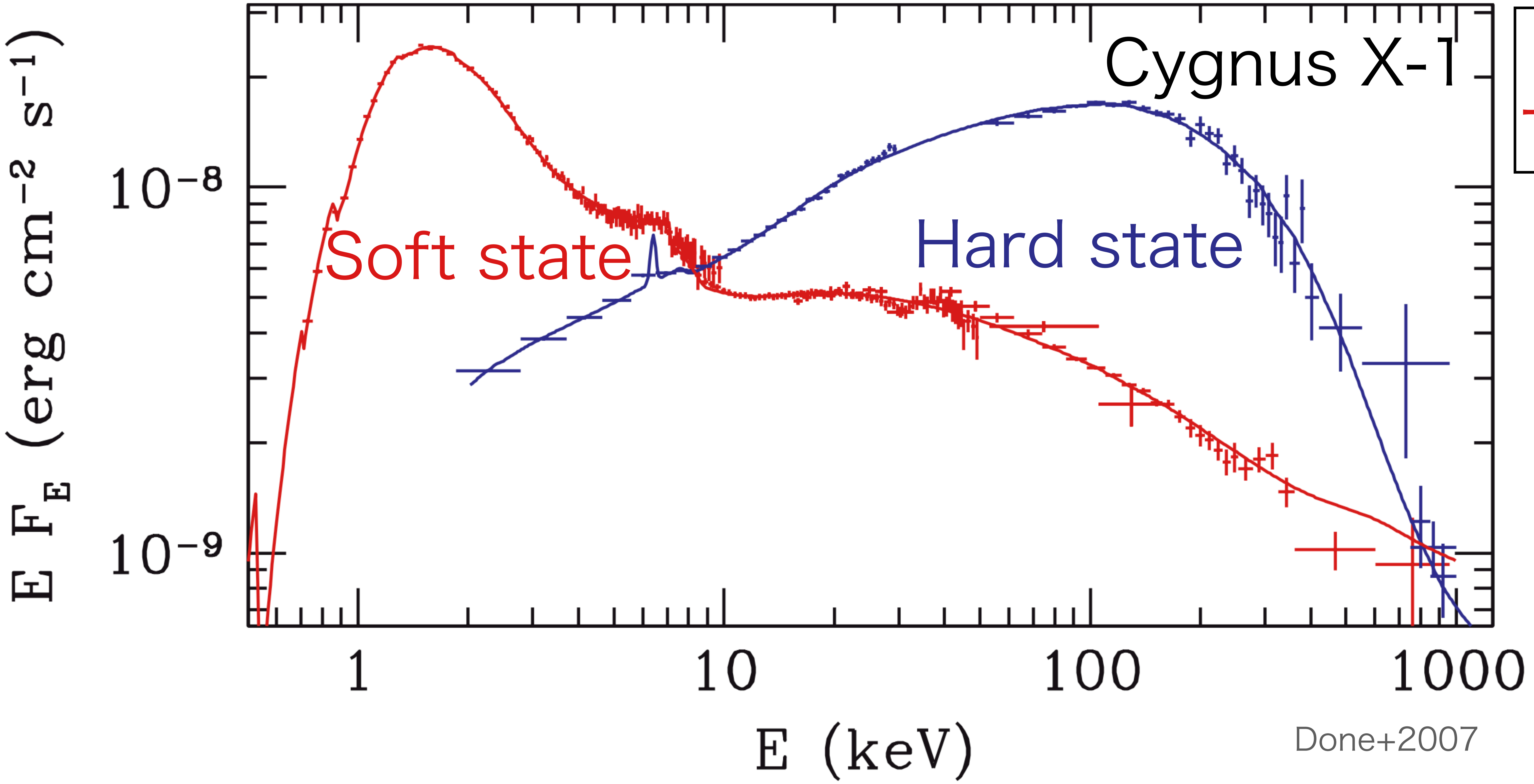
Gallo+2005



- Relativistic jet ejects from the central BH.
- Some BHXBs are observed in radio to gamma-rays.
- **Multi-wavelength emission mechanisms are still unknown.**

Multi-wavelength photon spectrum

BHXBs show the two-state in the photon spectrum.



Done+2007

Soft state

Thermal radiation (standard disk)

McConnell+2005

↕

Hard state

Thermal Inverse Compton

(Hot electron plasma; Corona)

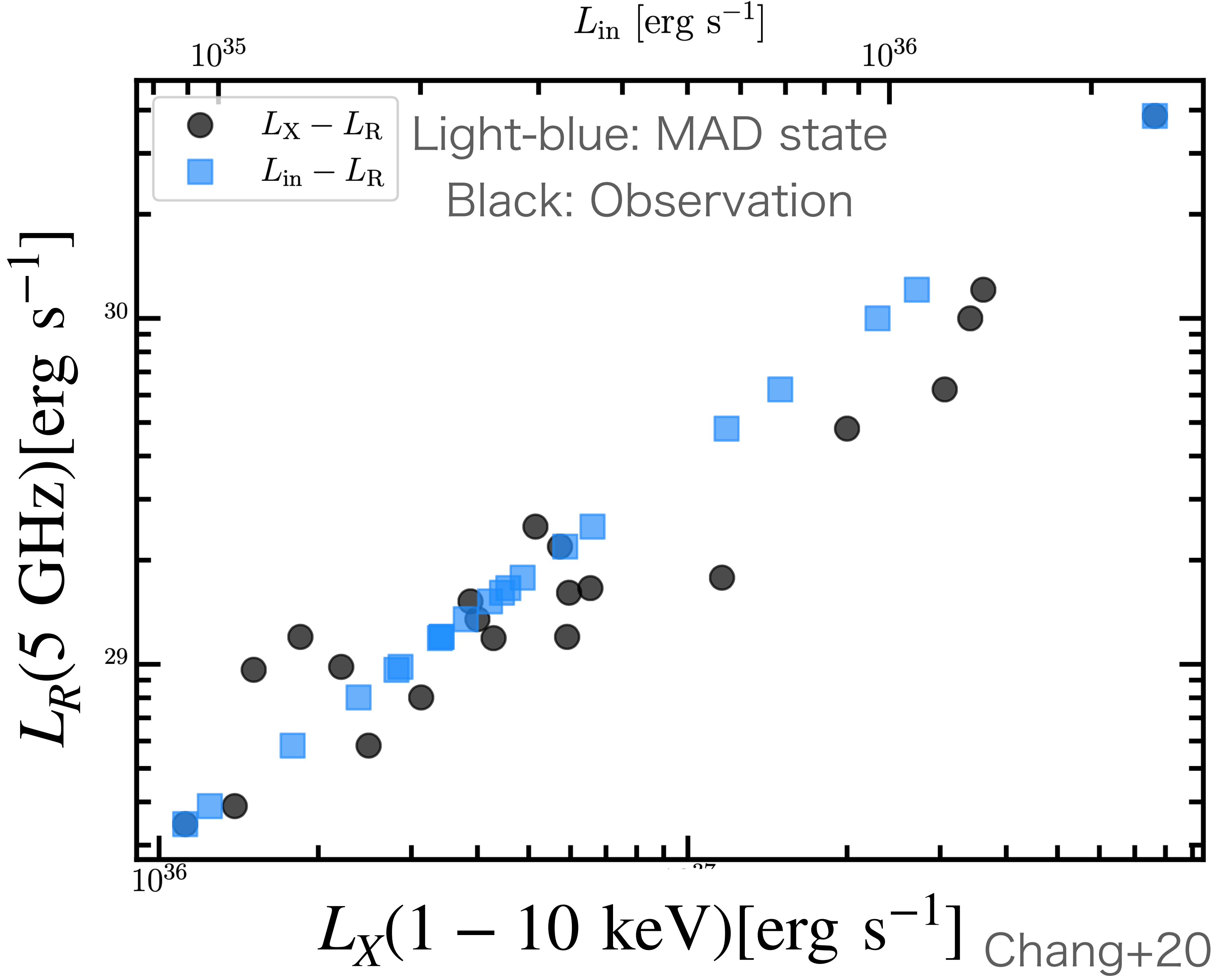
McConnell+2005

The origin of the corona is unknown.

We consider that the hot electron plasma comes from the hot accretion flows especially Magnetically Arrested Disk (MAD).

Magnetically Arrested Disks (MADs)

Radio and X-ray obs

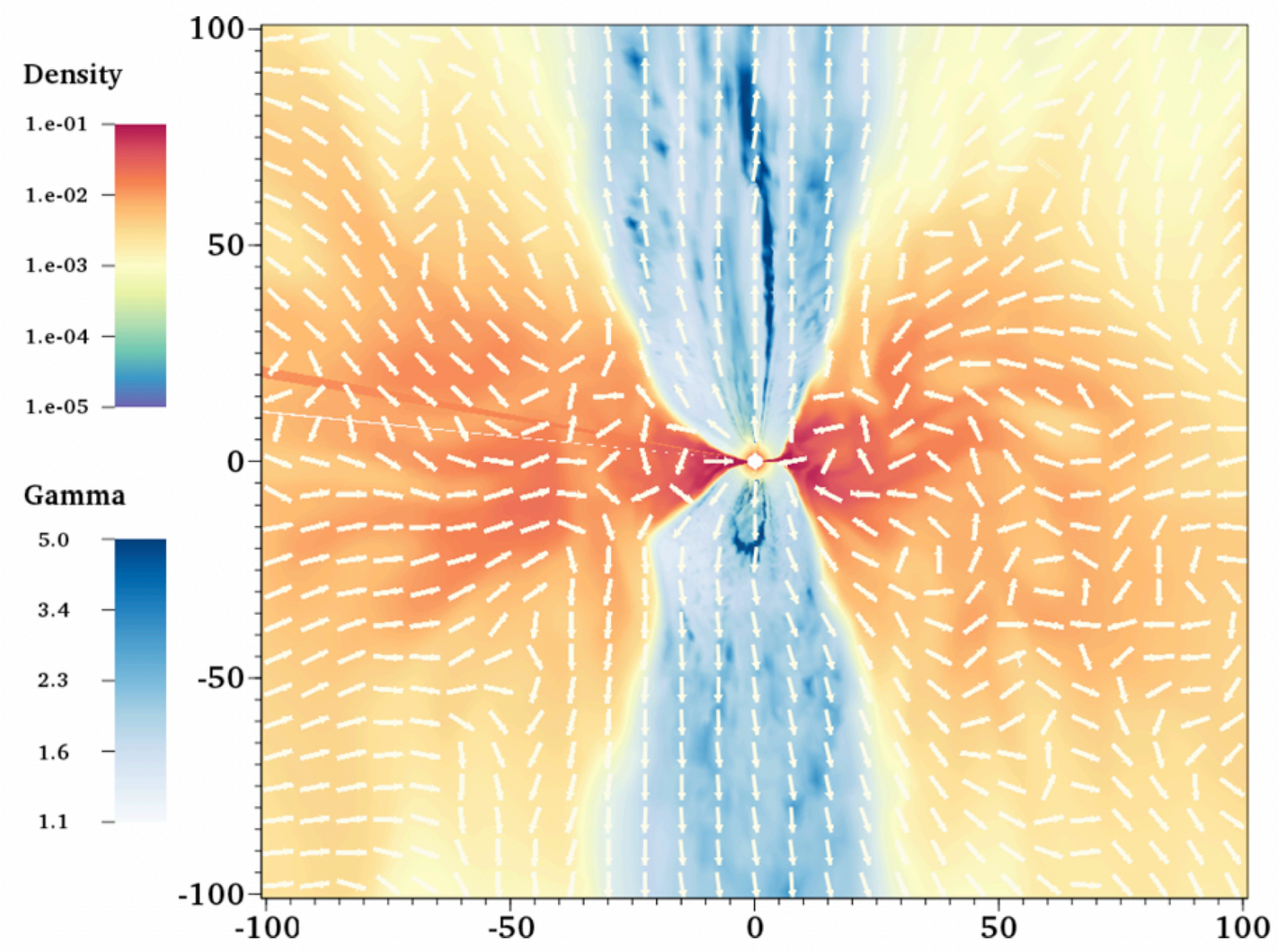


Light-blue: MAD state

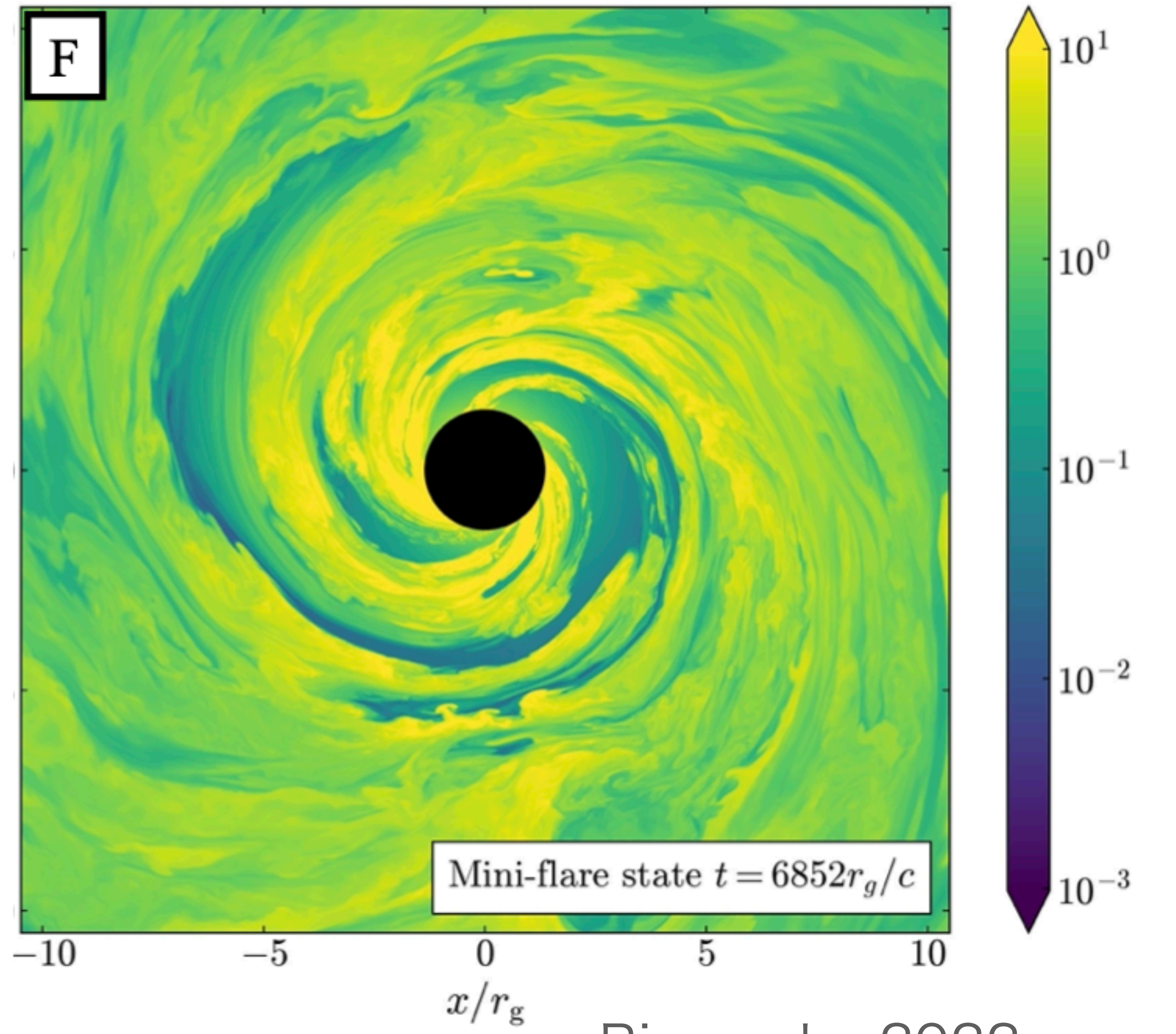
Black: Observation

Chang+2022

MHD simulations



Sadowski+2013



Ripperda+2022

Advantage of the MAD ▶

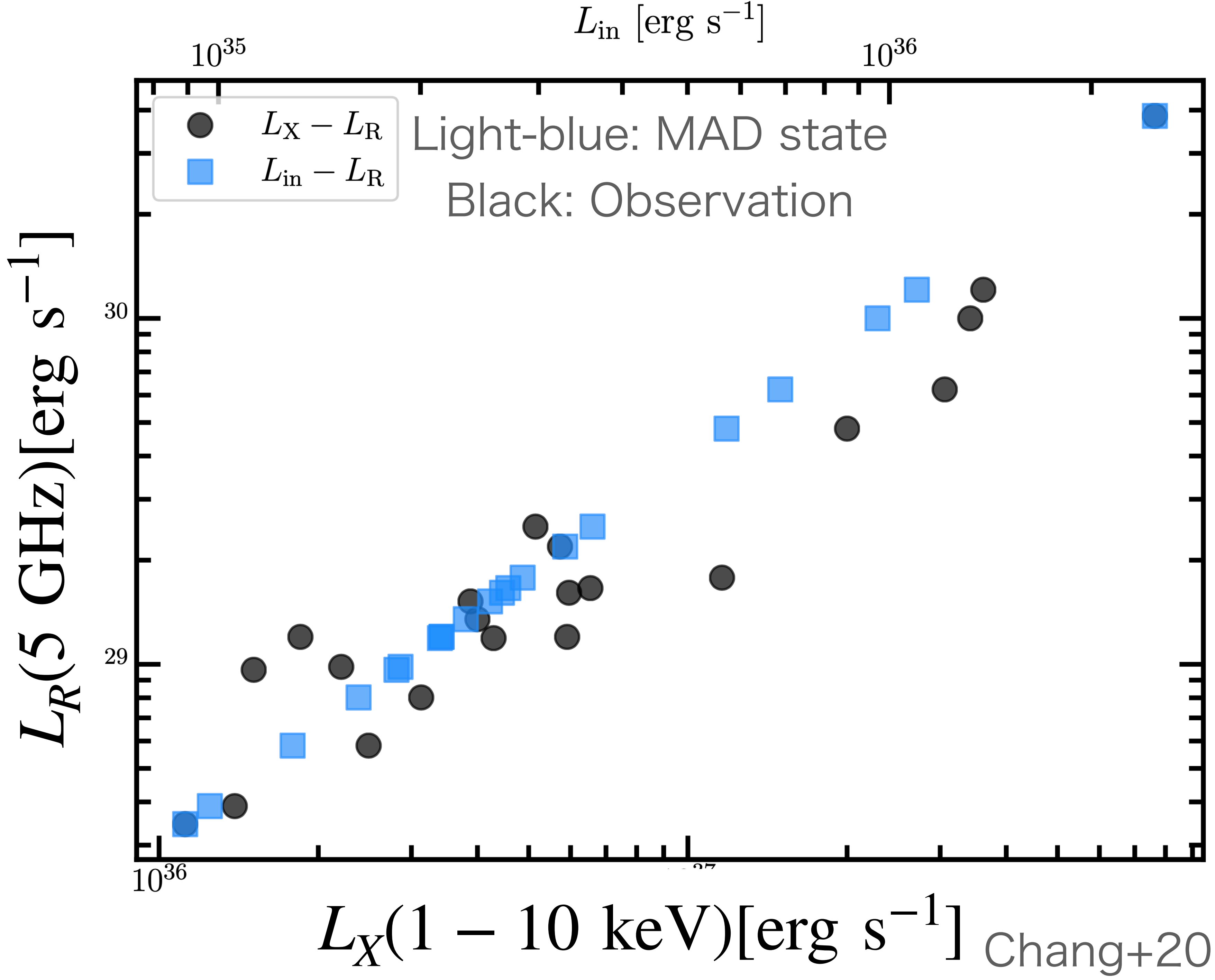
① Efficient jet production by the BZ process. Tchekhovskoy+2011

② Radio and X-ray luminosity relation supports the MAD state. Chang+2022

Particle acceleration and heating by the MHD turbulence is expected.

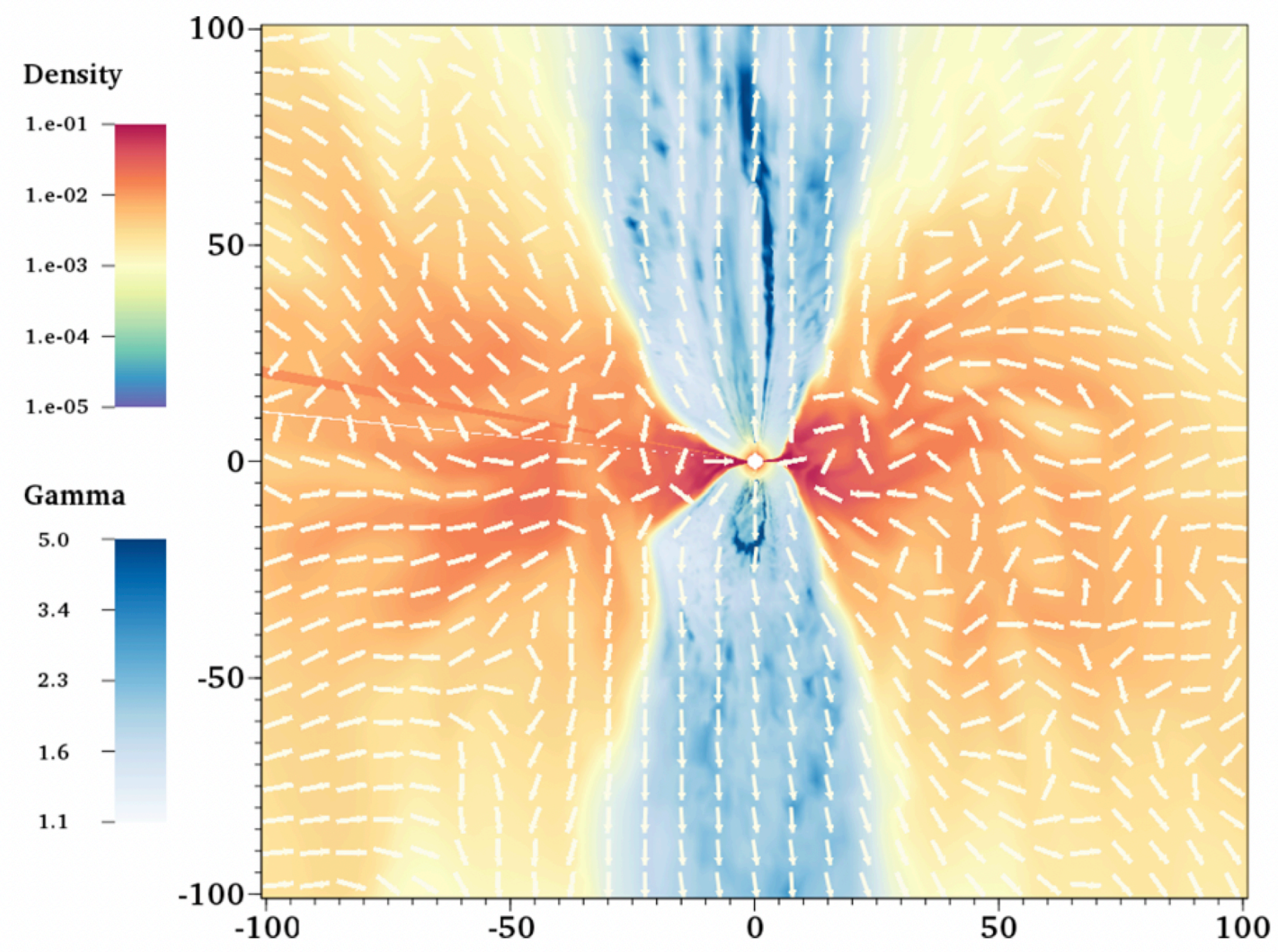
Magnetically Arrested Disks (MADs)

Radio and X-ray obs

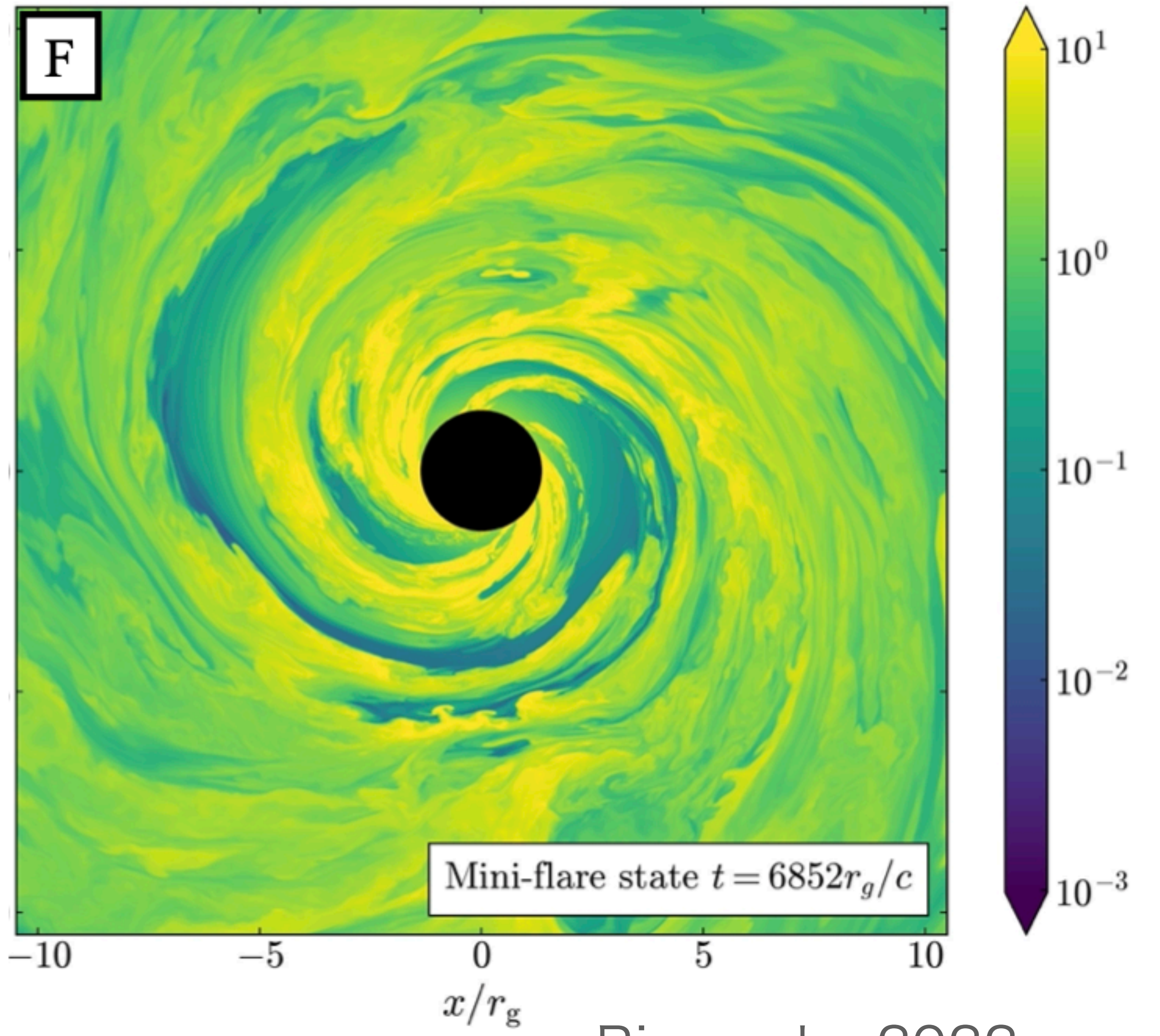


Chang+2022

MHD simulations



Sadowski+2013



Ripperda+2022

Advantage of the MAD ▶

① Efficient jet production by the BZ process. Tchekhovskoy+2011

② Radio and X-ray luminosity relation supports the MAD state. Chang+2022

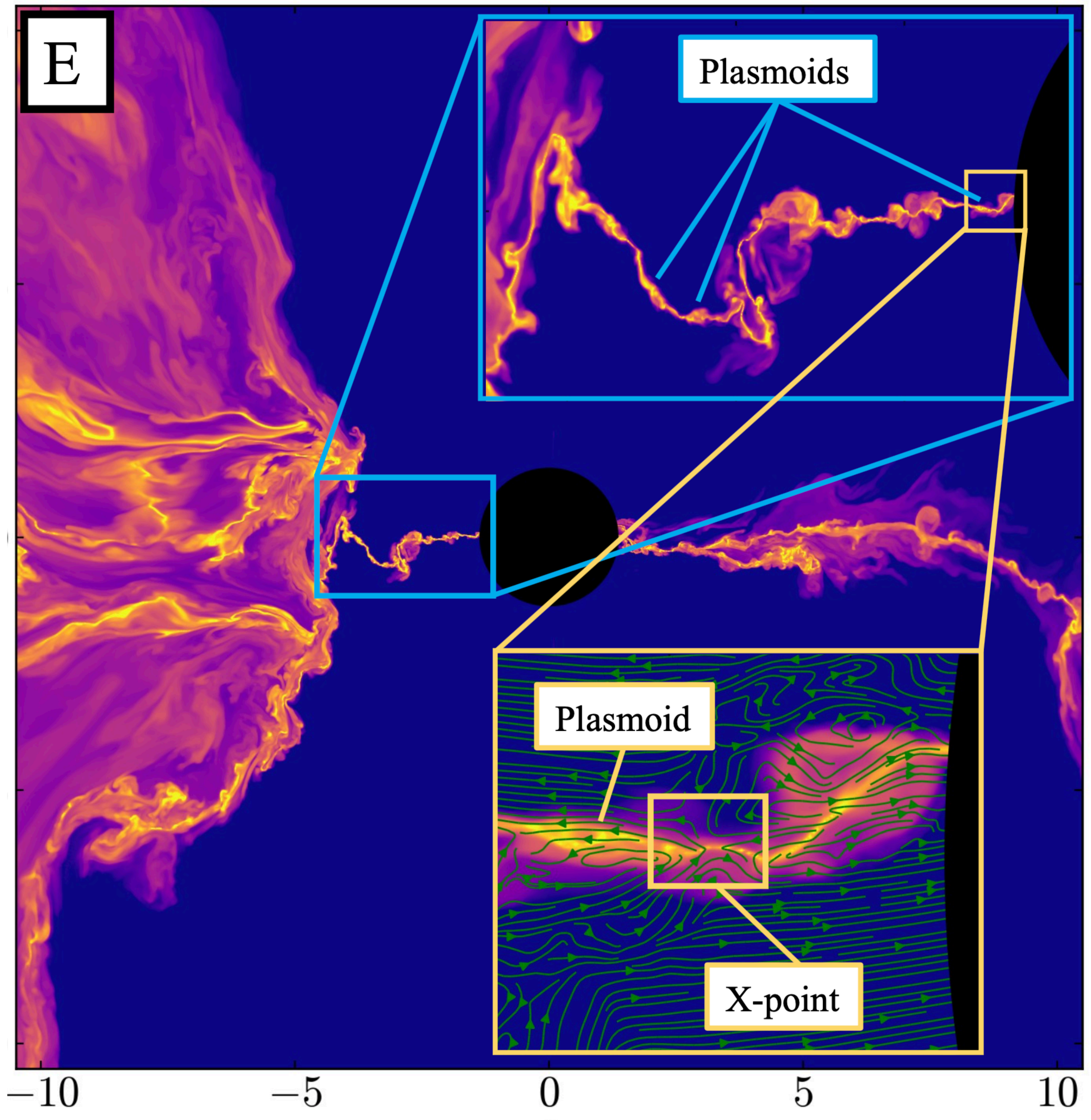
Remained Mystery:

How to inject the plasma into the jet?

Plasma injection model

Magnetic Reconnection in the vicinity of BH

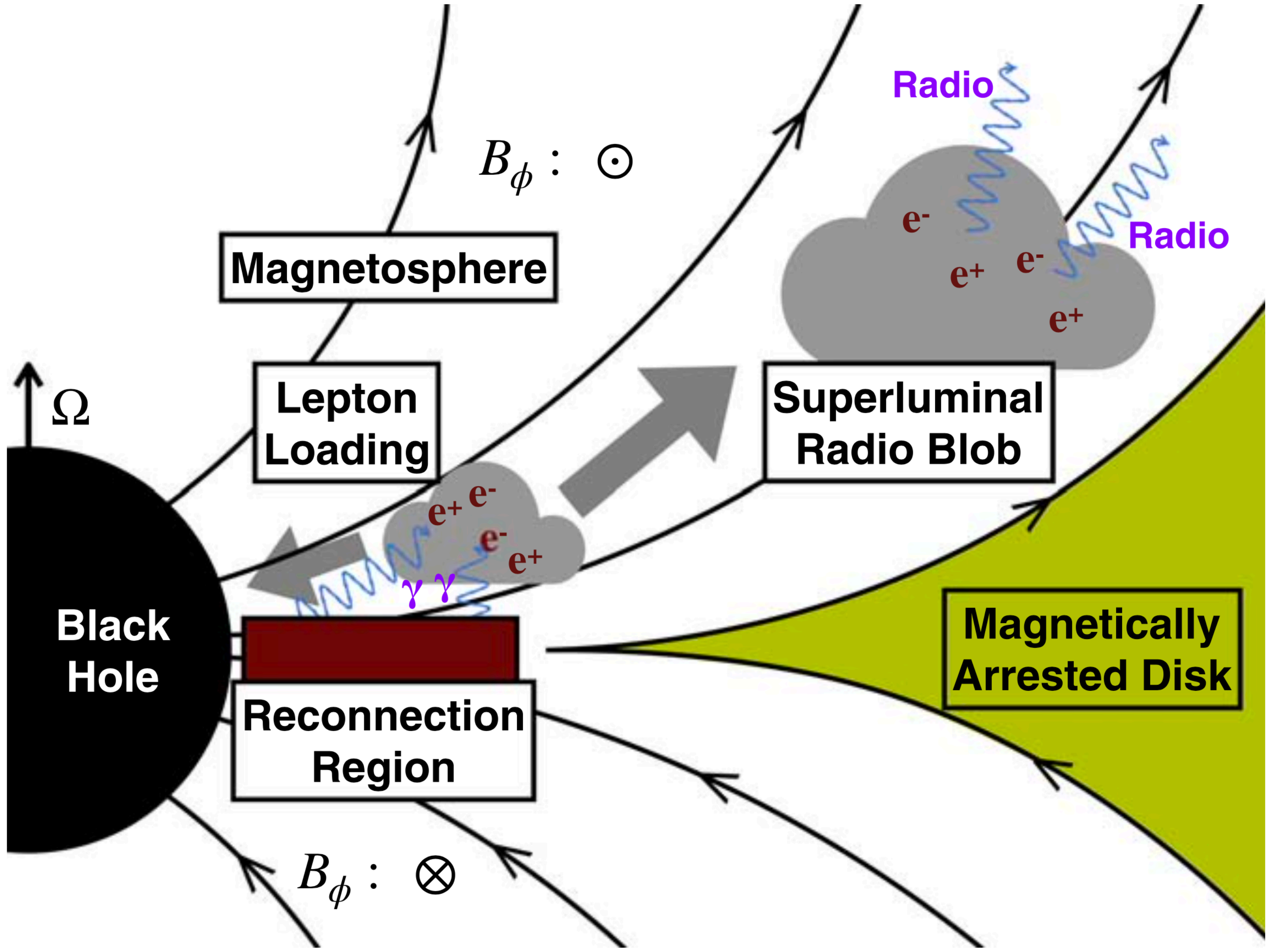
High resolution MHD simulation



Ripperda+2022

Particle Acceleration & Emission of HE photons

-> $\gamma\gamma \rightarrow e^+e^-$ pair production



Kimura+2022

Emitted jet material

Kimura+2022

-> e^+e^-pair plasma produced by $\gamma + \gamma \rightarrow e^+ + e^-$

Plasma injection model

Magnetic Reconnection

Particle Acceleration & Emission of HE photons

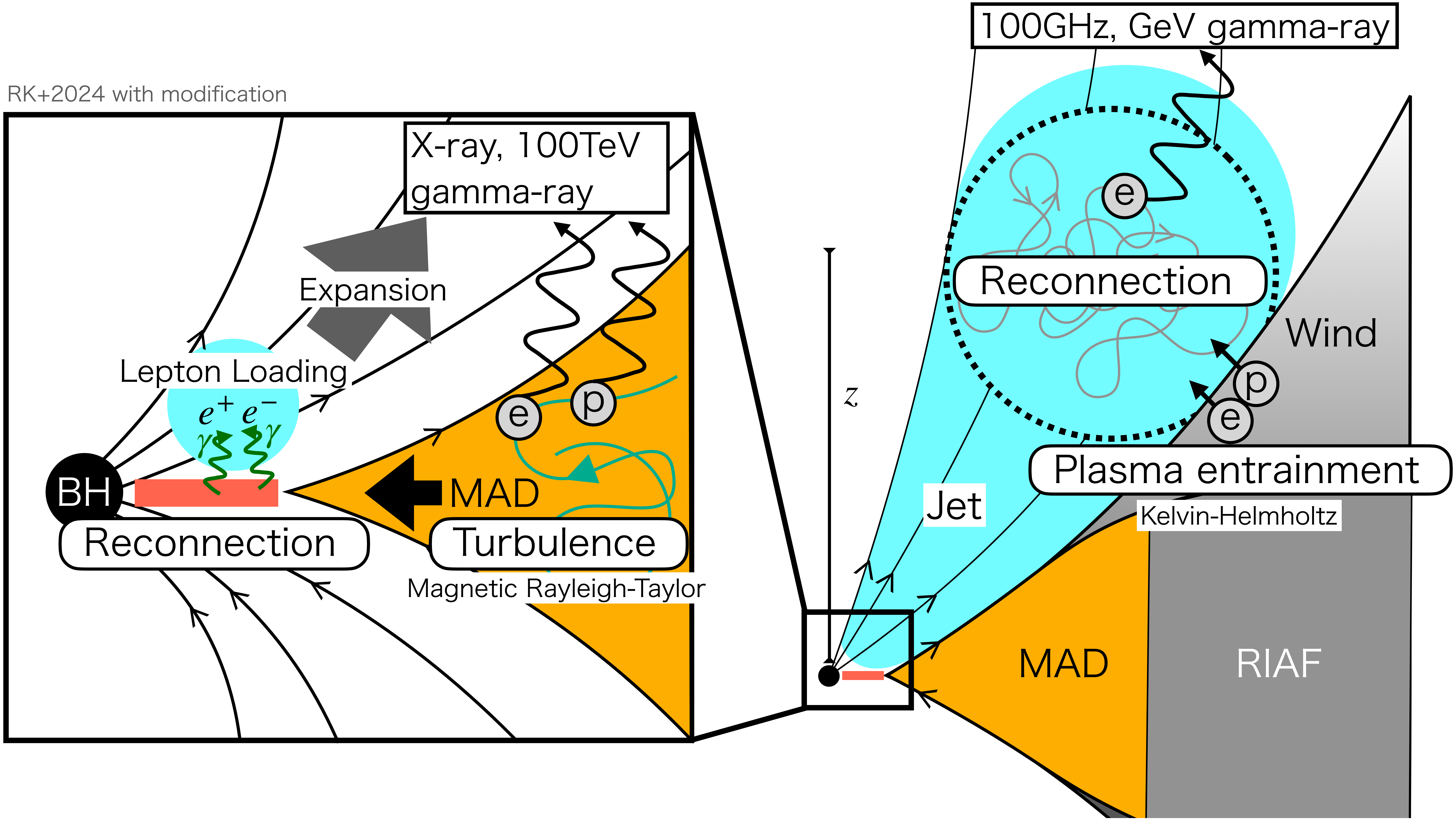
Our Study

We apply the emission model of jets and MADs with the particle injection model to Cygnus X-1.

-> e^+e^-pair plasma produced by $\gamma+\gamma \rightarrow e^+ + e^-$

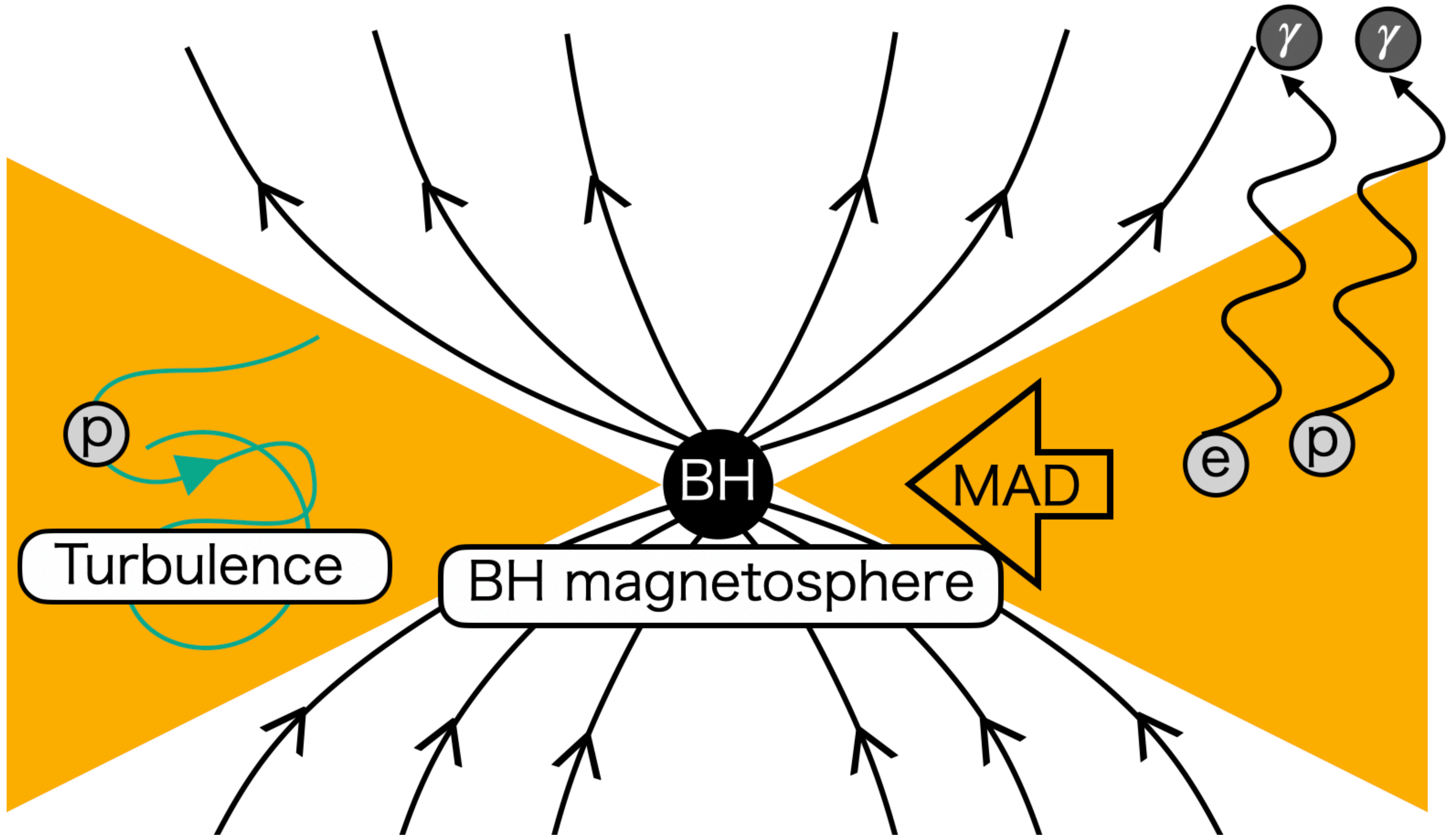
Overview of the Jet-MAD model

RK+2024 with modification

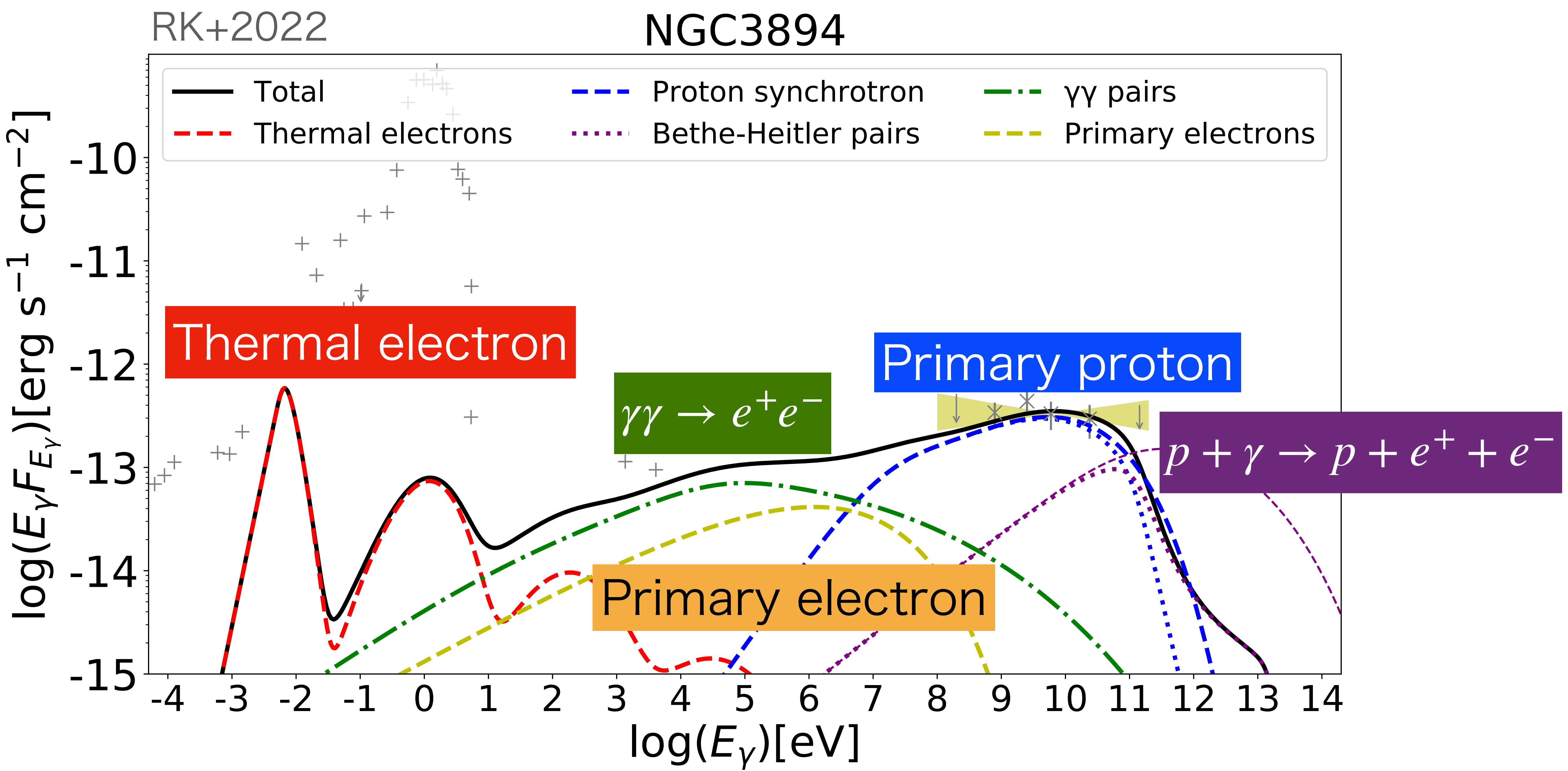


Emission from the MAD

Schematic image of MAD model



MHD turbulence accelerates the CRs.



Synchrotron radiation produce the broadband photons.

- 5 particle species

Thermal electron

Primary electron

Primary proton

$p+\gamma \to p+e^++e^-$

$\gamma+\gamma \to e^++e^-$

- Steady & one-zone approximation

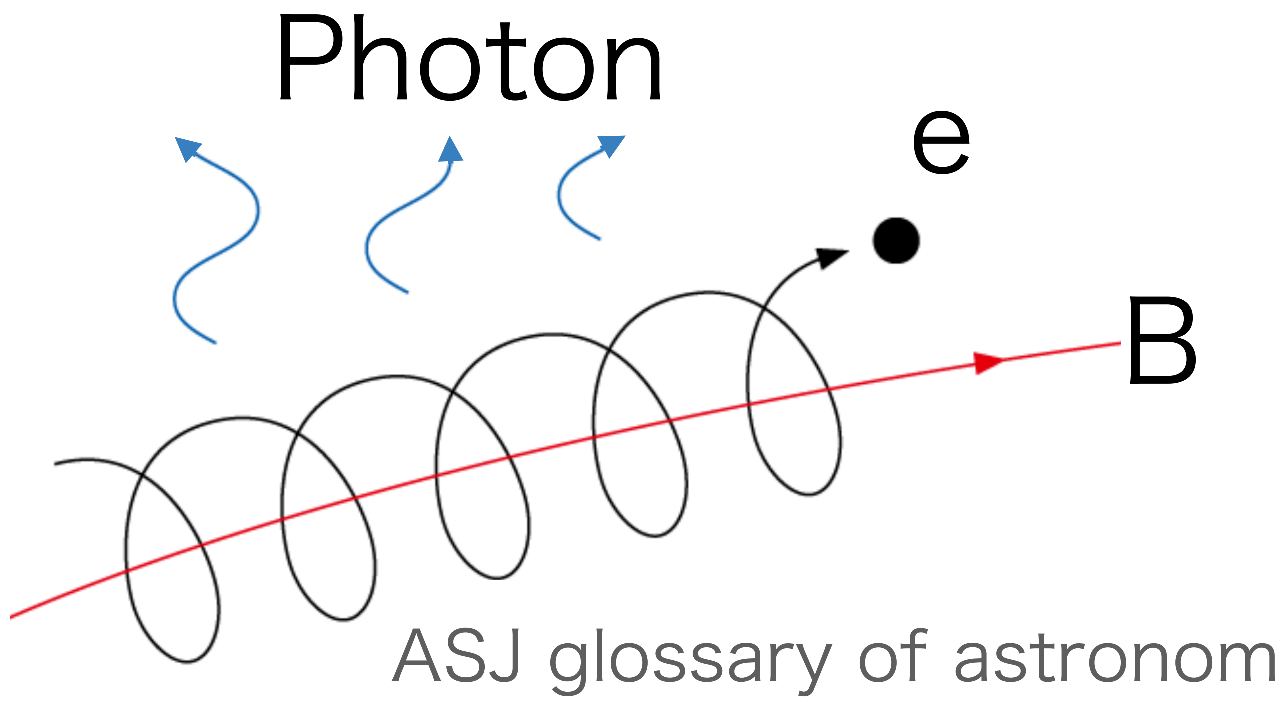
Transport Equation

$$-\underbrace{\frac{d}{dE_i}\left(\frac{N_{E_i}E_i}{t_{i,\mathrm{cool}}}\right)}_{\text{Cooling}} = \underbrace{\dot{N}_{E_i,\mathrm{inj}}}_{\text{Injection}} - \underbrace{\frac{N_{E_i}}{t_{\mathrm{esc}}}}_{\text{Escape}}$$

Physical processes in the MAD

■ Cooling time $t_{i,\mathrm{cool}}$

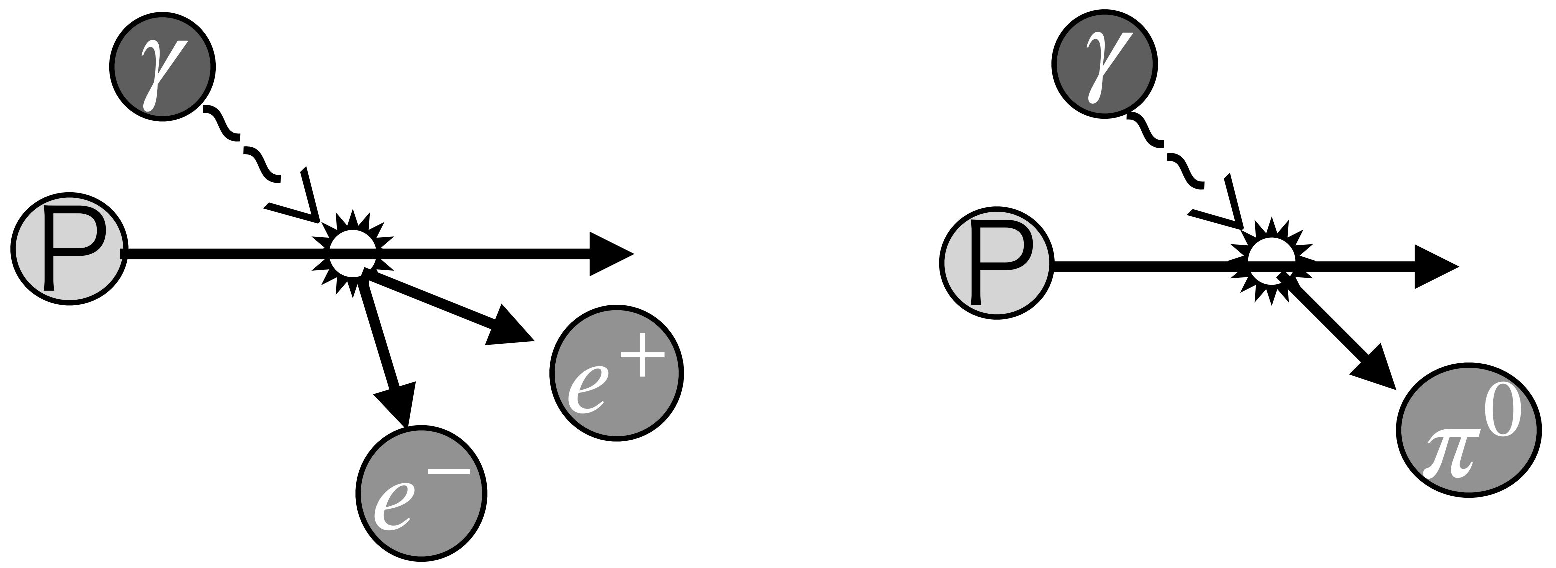
Synchrotron



ASJ glossary of astronomy

Proton photon interaction

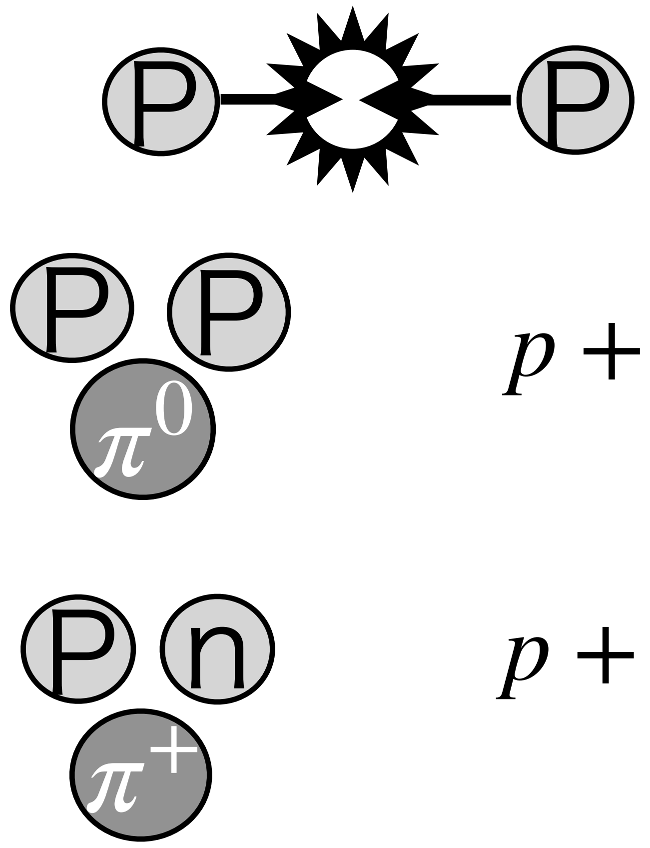
Bethe-Heitler



$p + \gamma \rightarrow p + e^{+} + e^{-}$

$p + \gamma \rightarrow p + \pi^{0}$

pp collision



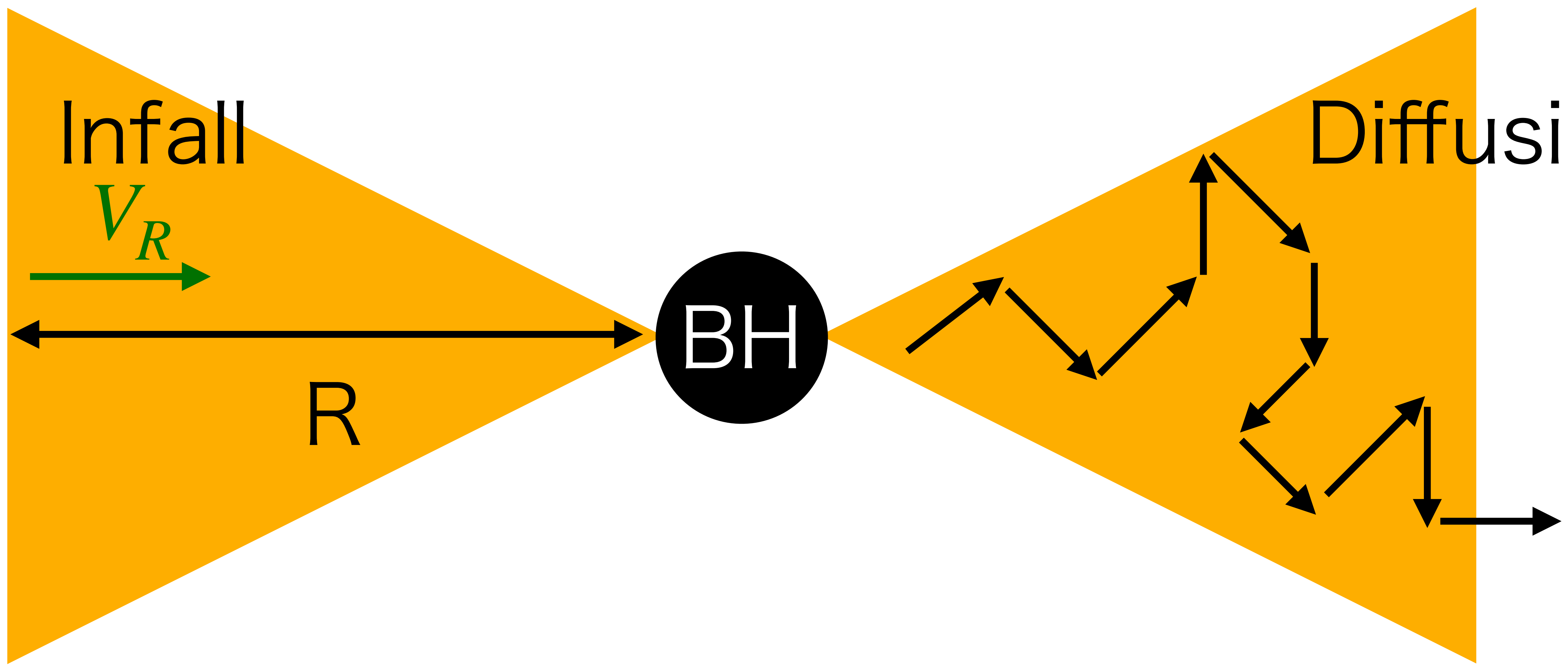
$p + p \rightarrow p + p + \pi^{0}$

$p + p \rightarrow p + n + \pi^{+}$

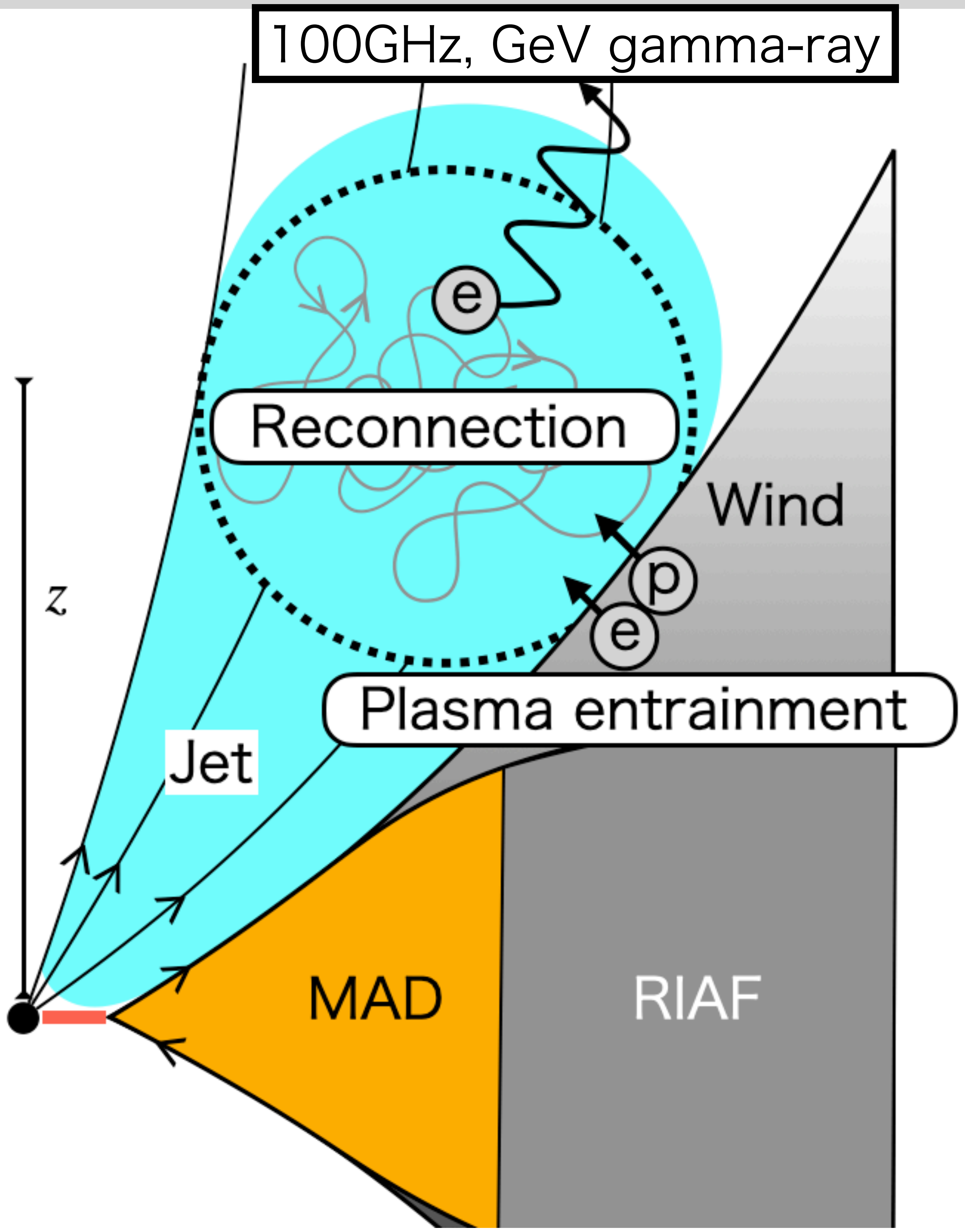
■ Escape time t_{esc}

(i)Infall time

(ii)Diffusion time

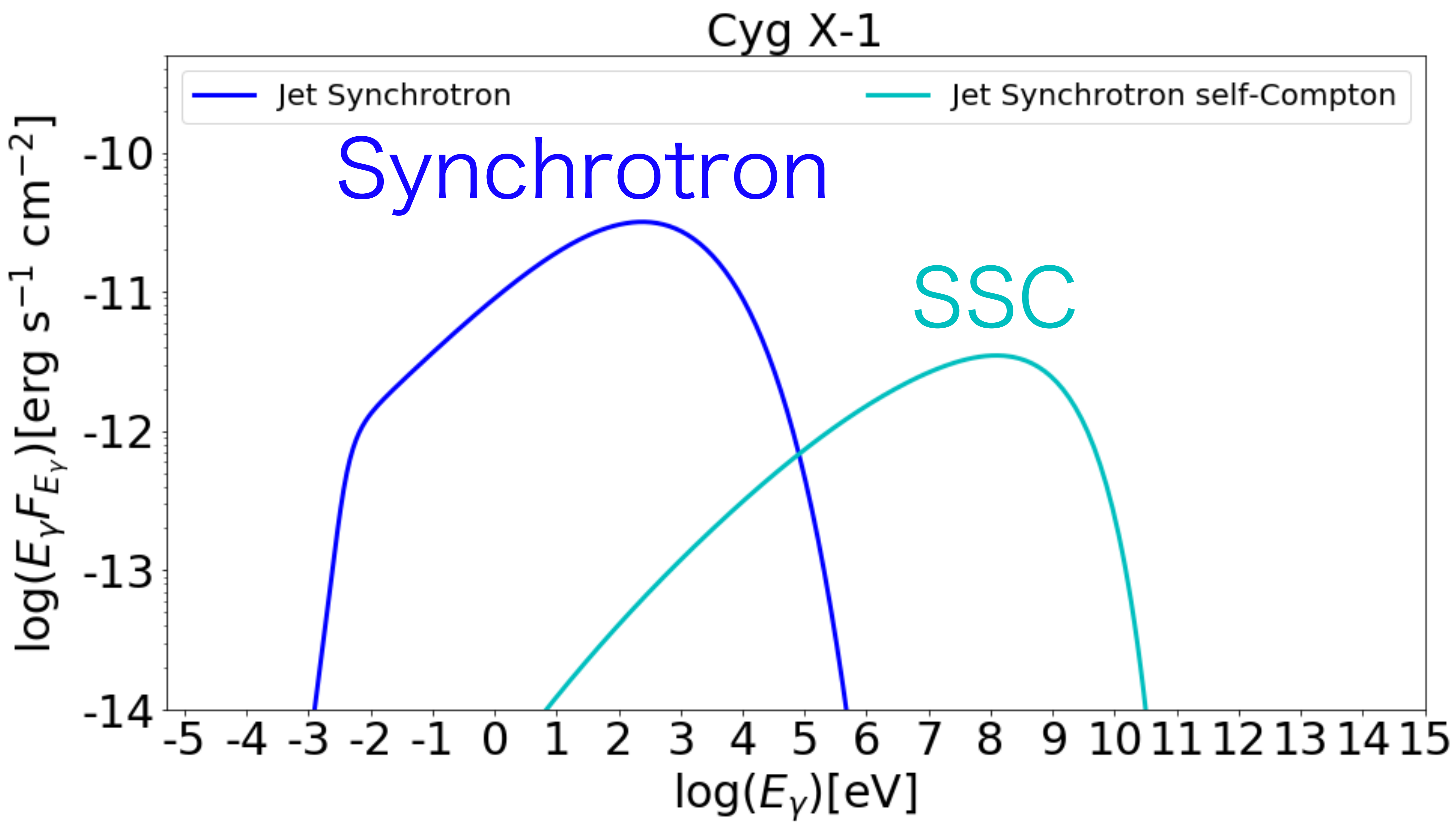


Emission from the Jet



Magnetic reconnection inside the jet accelerates the particles.

- Steady & one-zone approximations
- Nonthermal Electrons



Accelerated electrons emit multi-wavelength photons via synchrotron and synchrotron-self Compton.

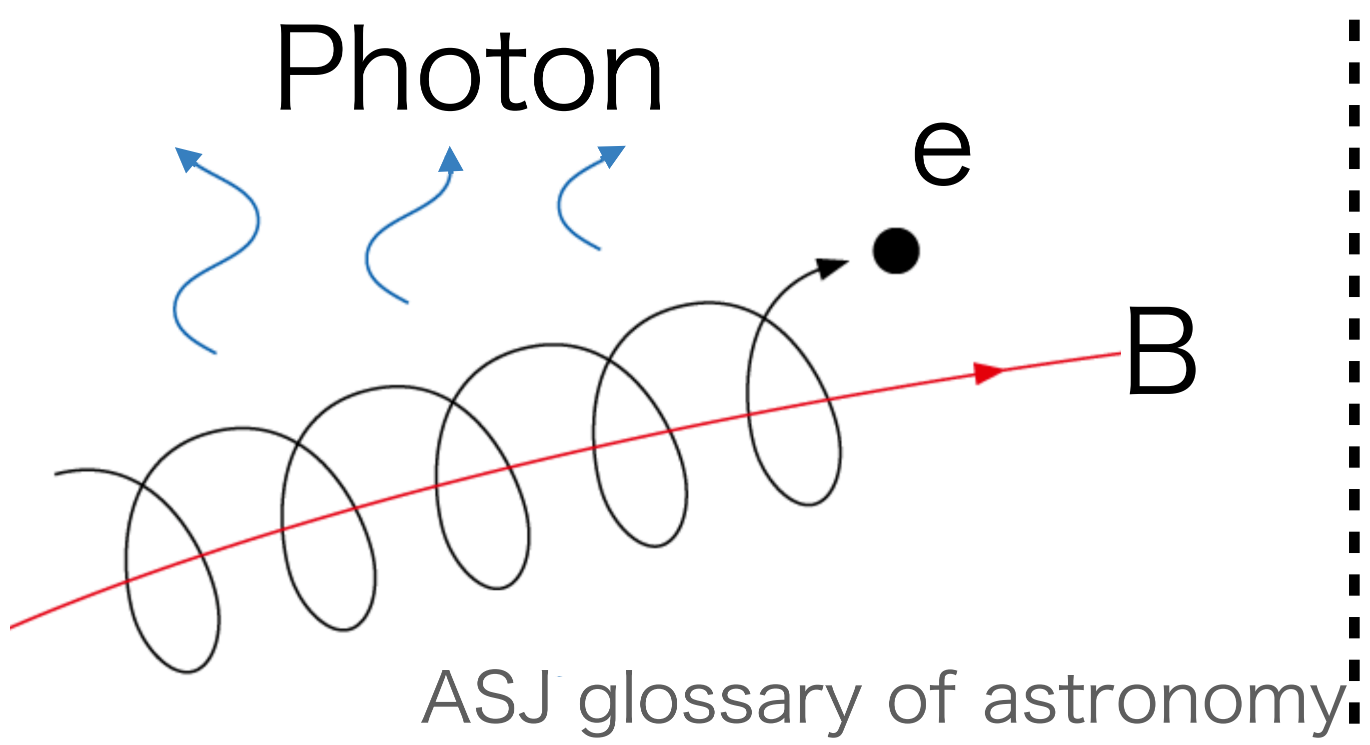
Transport Equation

$$\boxed{\underbrace{-\frac{d}{dE_e}\left(\frac{N_{E_e}E_e}{t_{e,\mathrm{cool}}}\right)}_{\text{Cooling}} = \underbrace{\dot{N}_{E_e,\mathrm{inj}}}_{\text{Injection}} - \underbrace{\frac{N_{E_e}}{t_{\mathrm{esc}}}}_{\text{Escape}}}$$

Physical processes in the jet

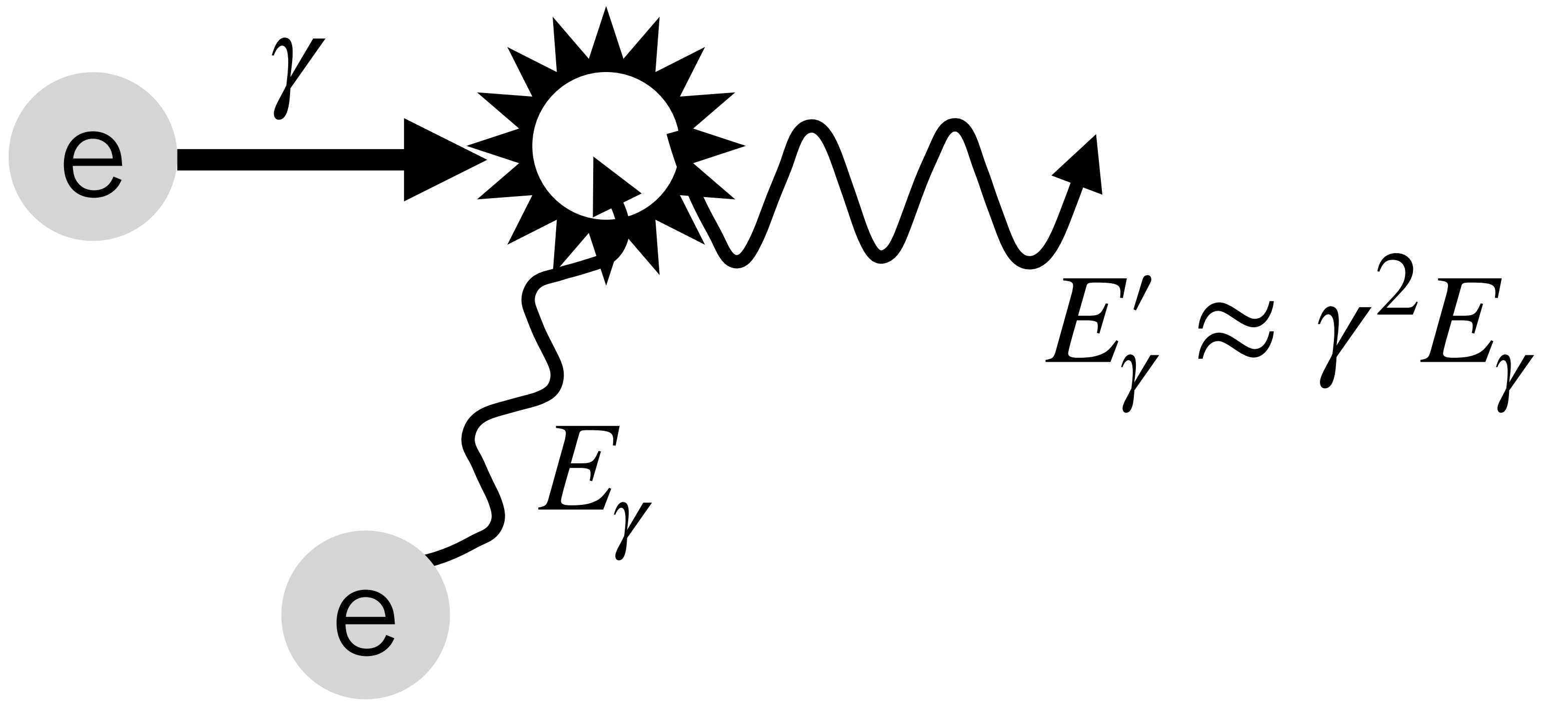
■ Cooling time $t_{i,\mathrm{cool}}$

Synchrotron

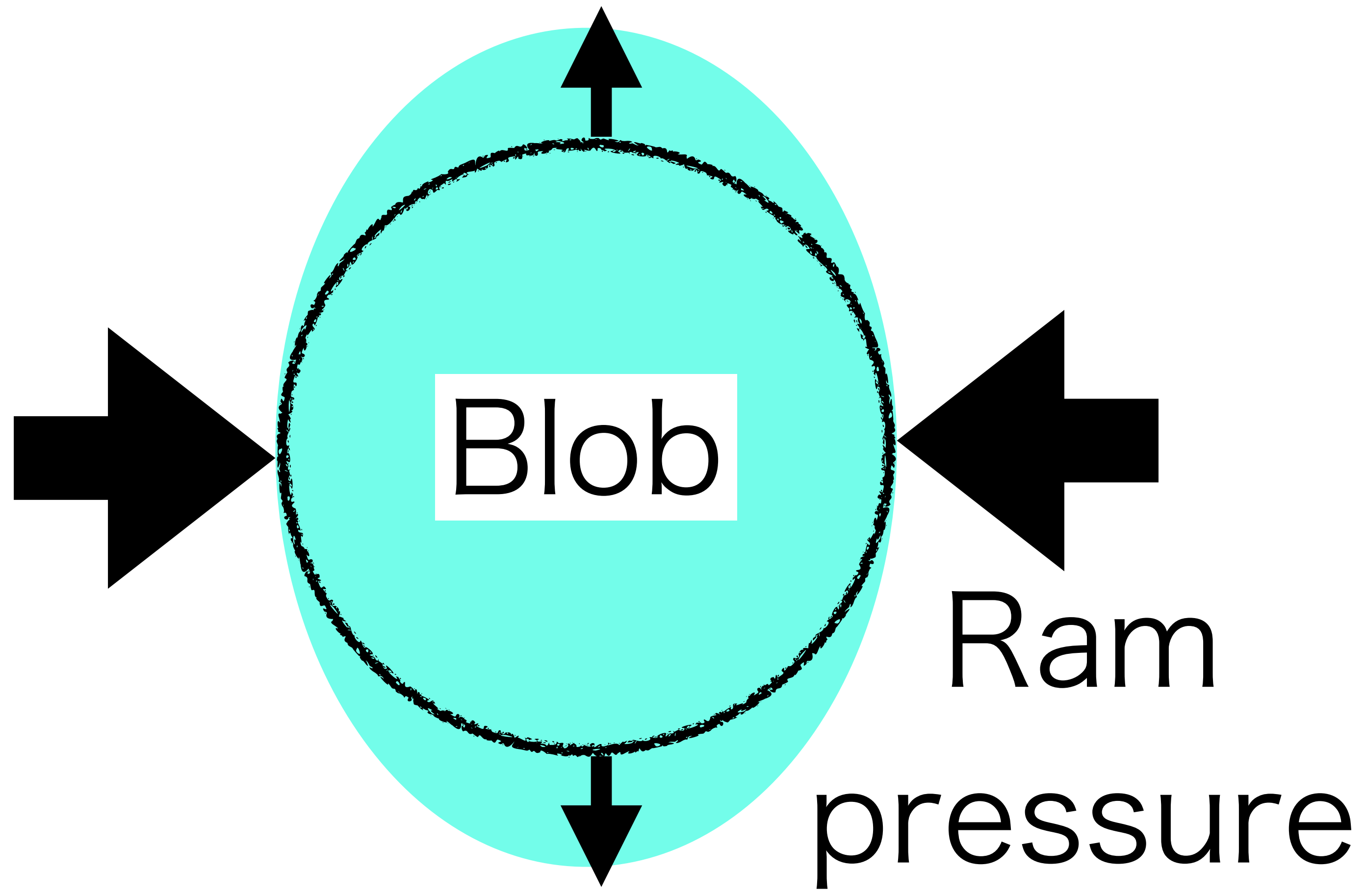


ASJ glossary of astronomy

Synchrotron Self-Compton

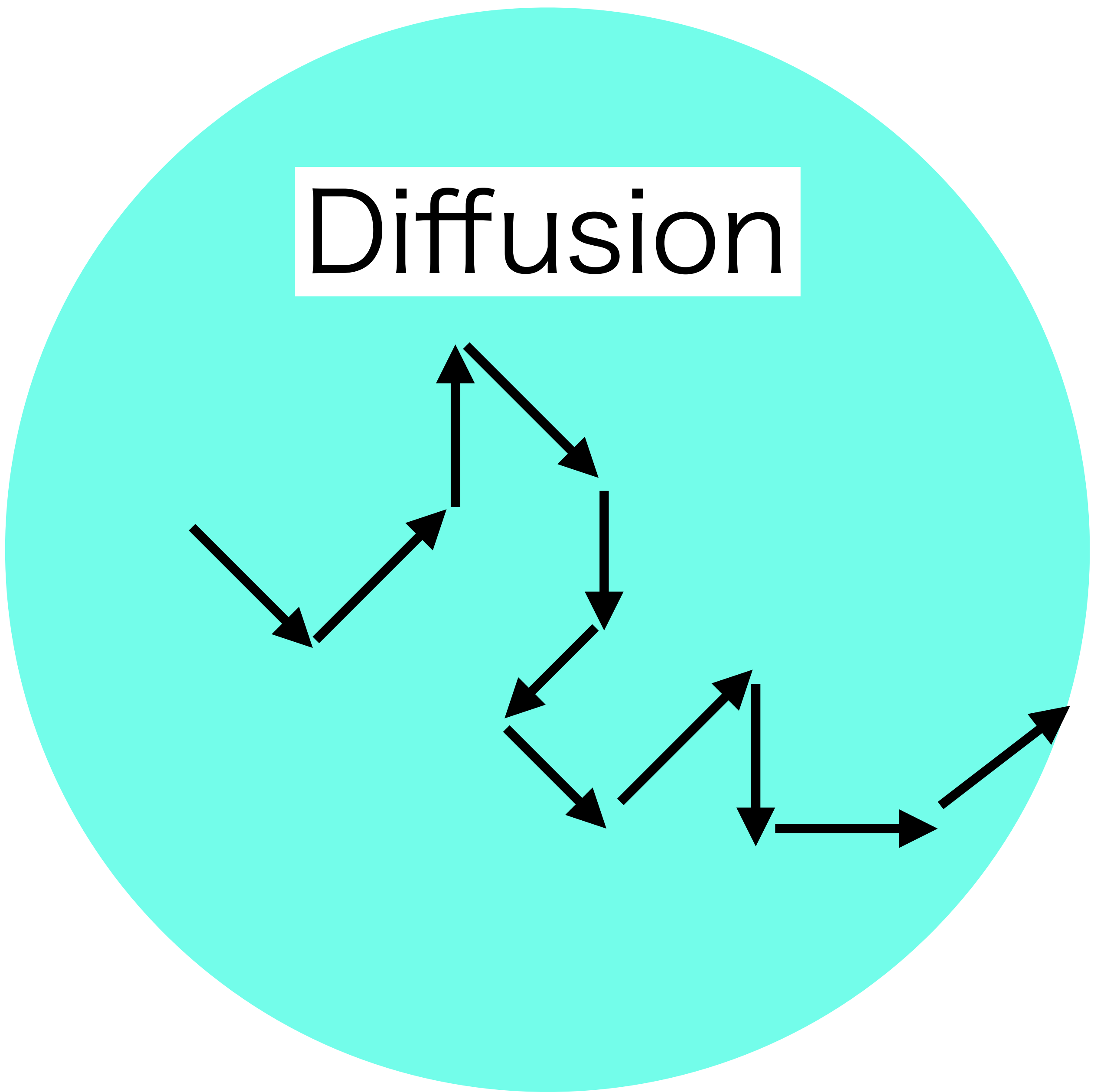


Adiabatic expansion



■ Escape time t_{esc}

(i) Diffusion time



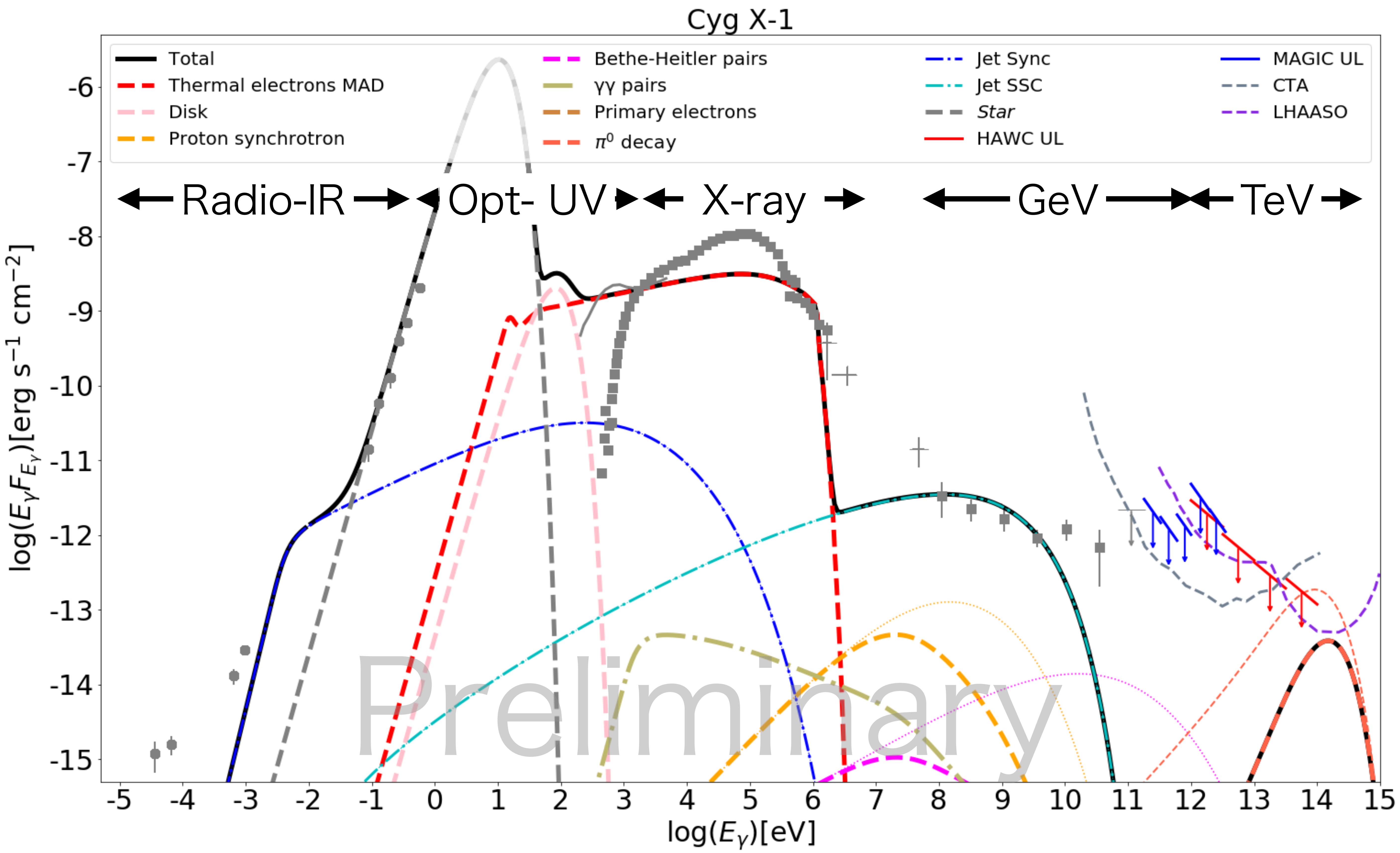
Result: Photon spectrum

Parameter：$\sigma_{\mathrm{ent}} \sim 0.001,\ \dot{m} = 0.1$

$B_{\mathrm{jet}} \simeq 6.0 \times 10^2\ \mathrm{G}$

$\sigma \sim 1.1 \times 10^2 \quad \delta_D = 2.1$

$s_{\mathrm{inj,MAD}} = 1.1,\ s_{\mathrm{inj,jet}} = 2.1$

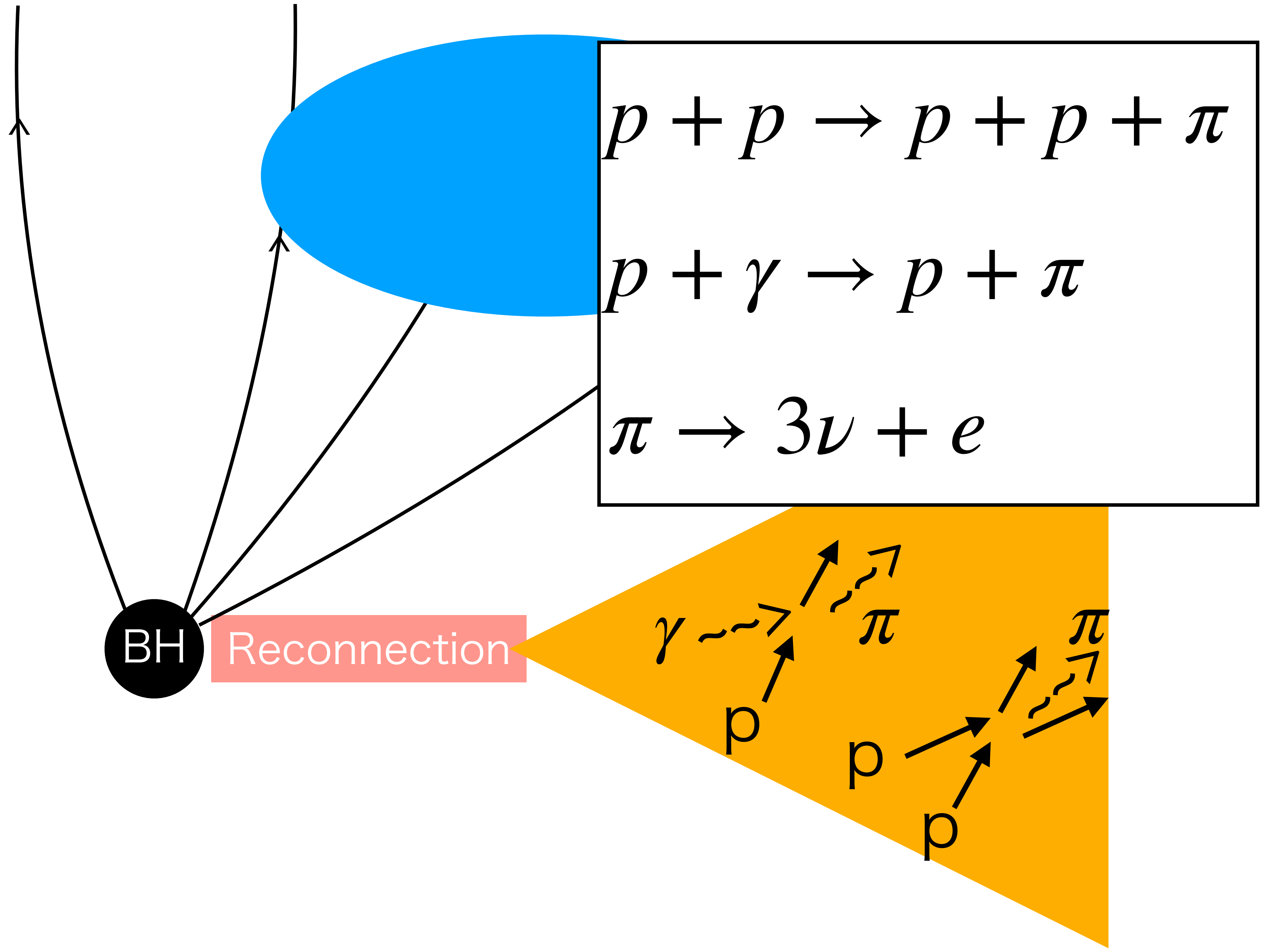


	Emission mechanisms
Radio	Synchrotron of jet
IR-Opt	Thermal radiation of star
X-ray	Thermal Comptonization of the MAD + **Other components**
GeV	SSC of jet
100TeV	Pion decay of the MAD (pp inelastic collision, photomeson production)

The Jet-MAD model can roughly explain the multi-wavelength observational data.

Neutrino Emission from MAD

Schematic image of the neutrino emission from MADs



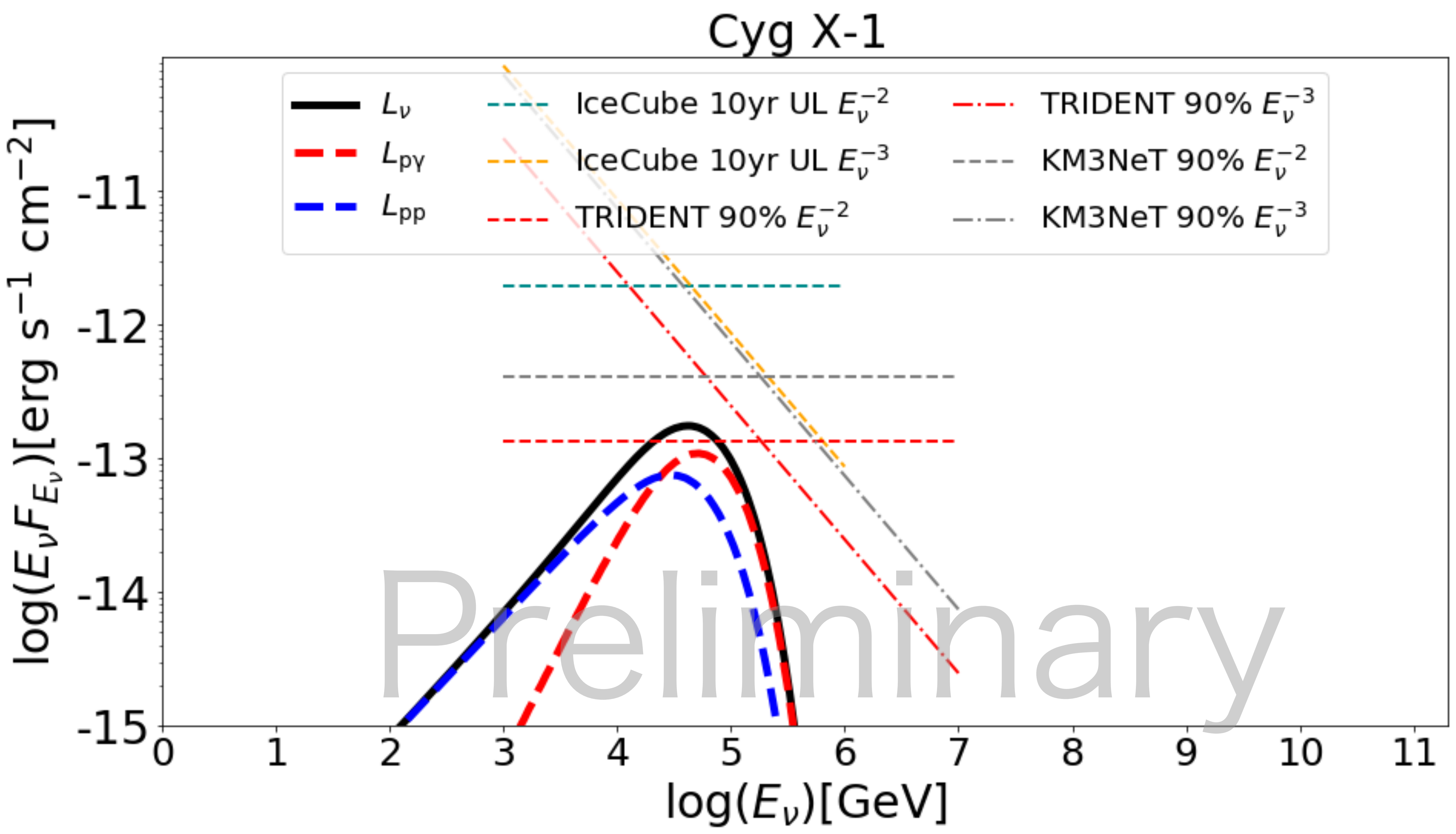
$p + p \rightarrow p + p + \pi$

$p + \gamma \rightarrow p + \pi$

$\pi \rightarrow 3\nu + e$

We consider the neutrino emission from the pp collision ($p + p \rightarrow p + p + \pi$) and the photomeson production ($p + \gamma \rightarrow p + \pi$).

Neutrino emission from **MADs**



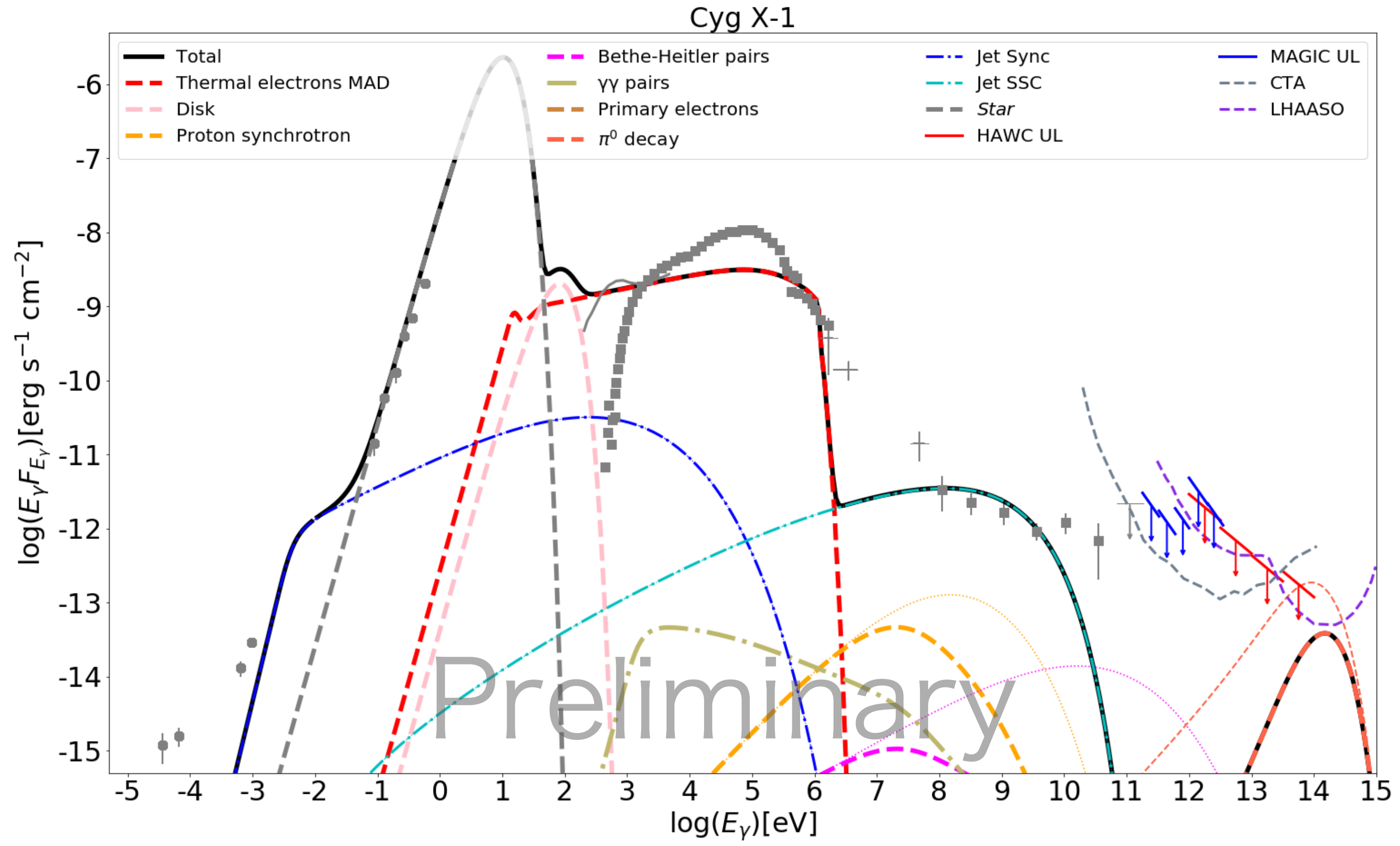
Proton maximum energy is 10^{15} eV

-> Neutrino energy is $\approx 10^{14}$ eV.

IceCube-Gen2 and TRIDENT may detect the neutrino emission from MAD.

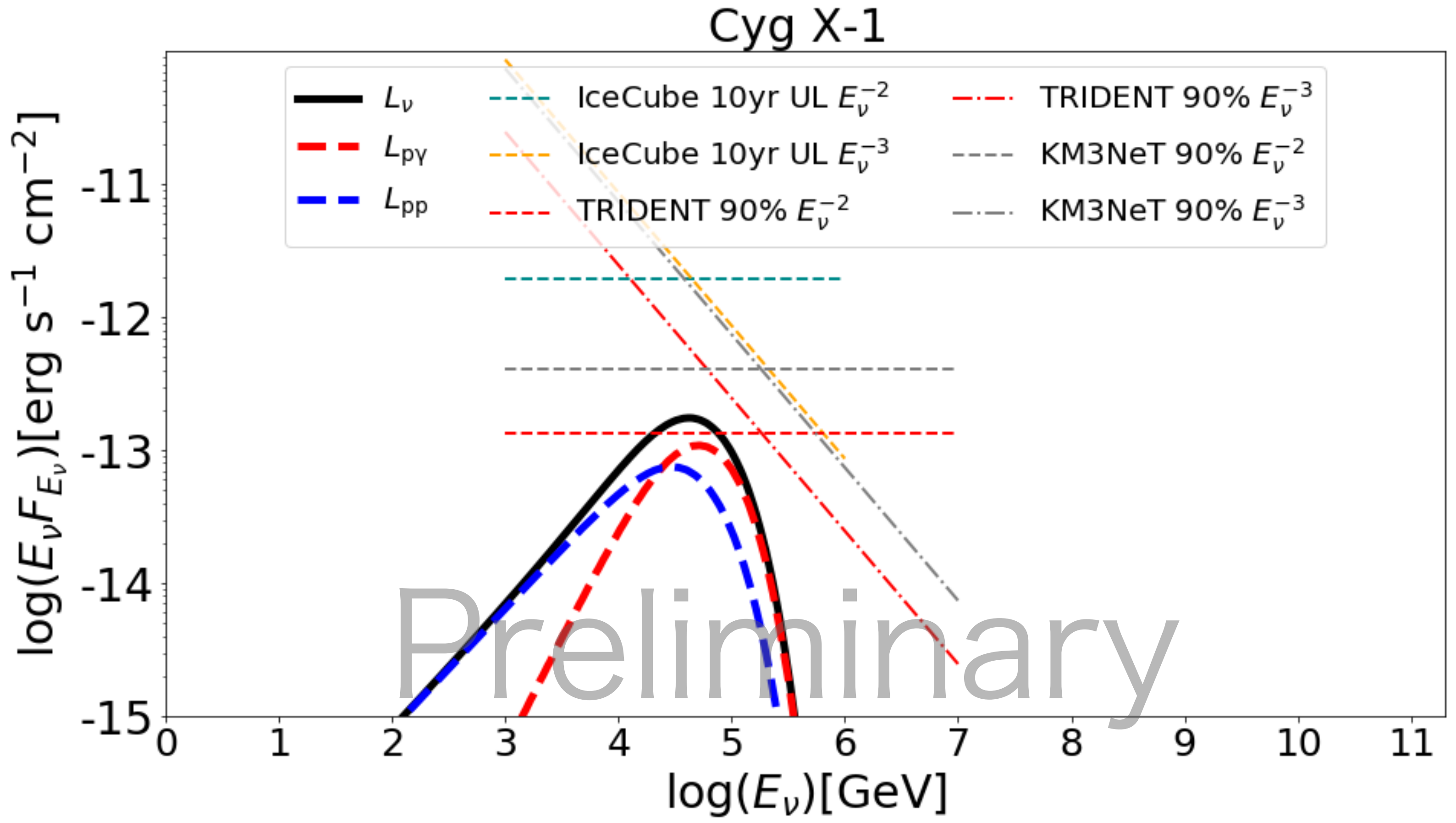
Summary

- ✓ We apply the multi-wavelength emission model **with the particle injection scenario** to **Cygnus X-1.**
- ✓ **The Jet-MAD model can roughly explain the observational data of Cygnus X-1.**
  - ■ **Radio: Electron synchrotron (Jet)**
  - ■ **X-ray: Thermal electron (MAD)**
    + Other components
  - ■ **Gamma-ray: Electron IC (Jet) (<TeV)**
    +Pion decay (MAD) (>100 TeV)



- ✓ **To explain the X-rays perfectly, we need other components.**
- ✓ We estimate neutrino emission from Jet-MAD model

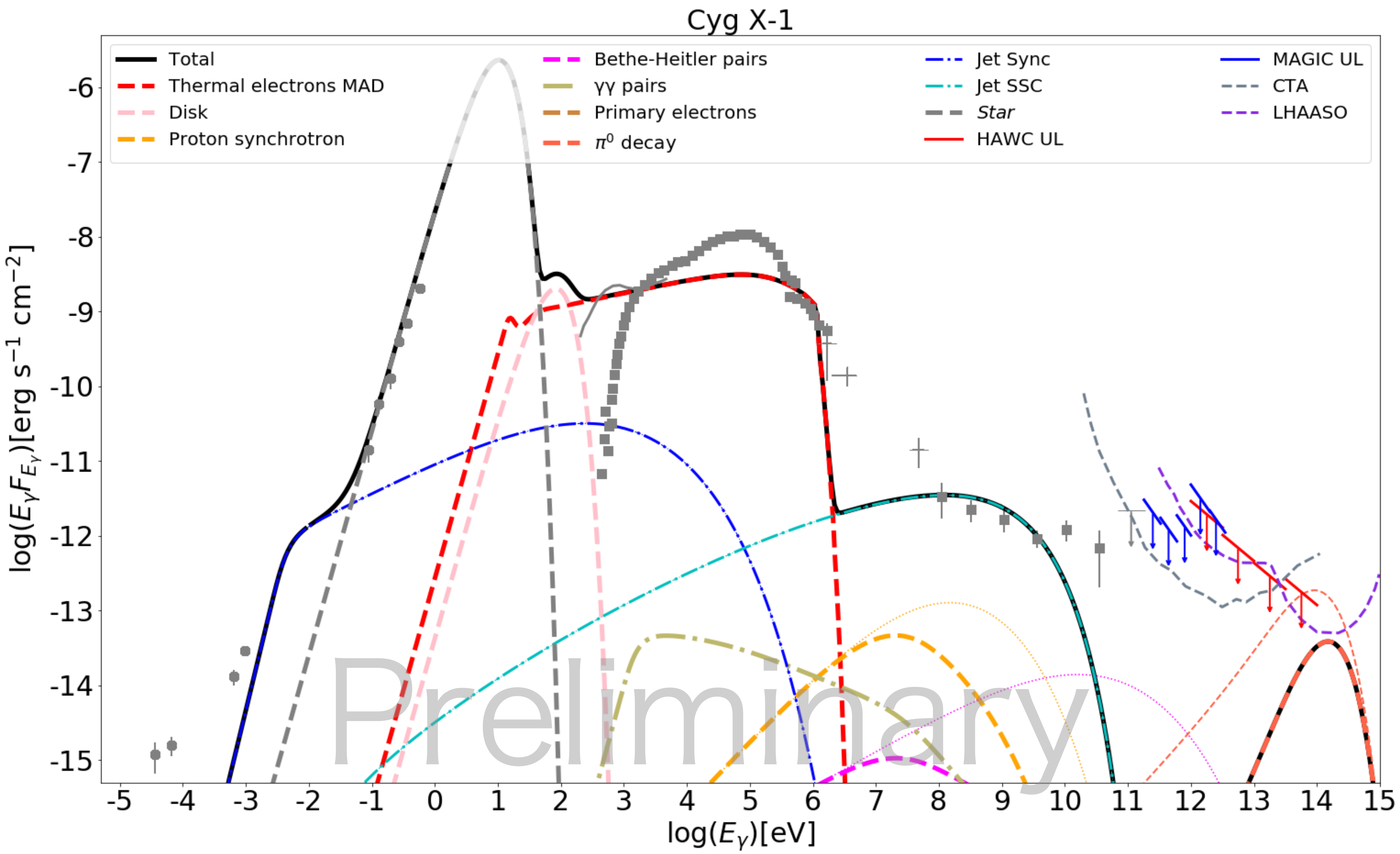
↳ **IceCube-Gen2 and TRIDENT may detect the neutrino emission from MAD.**

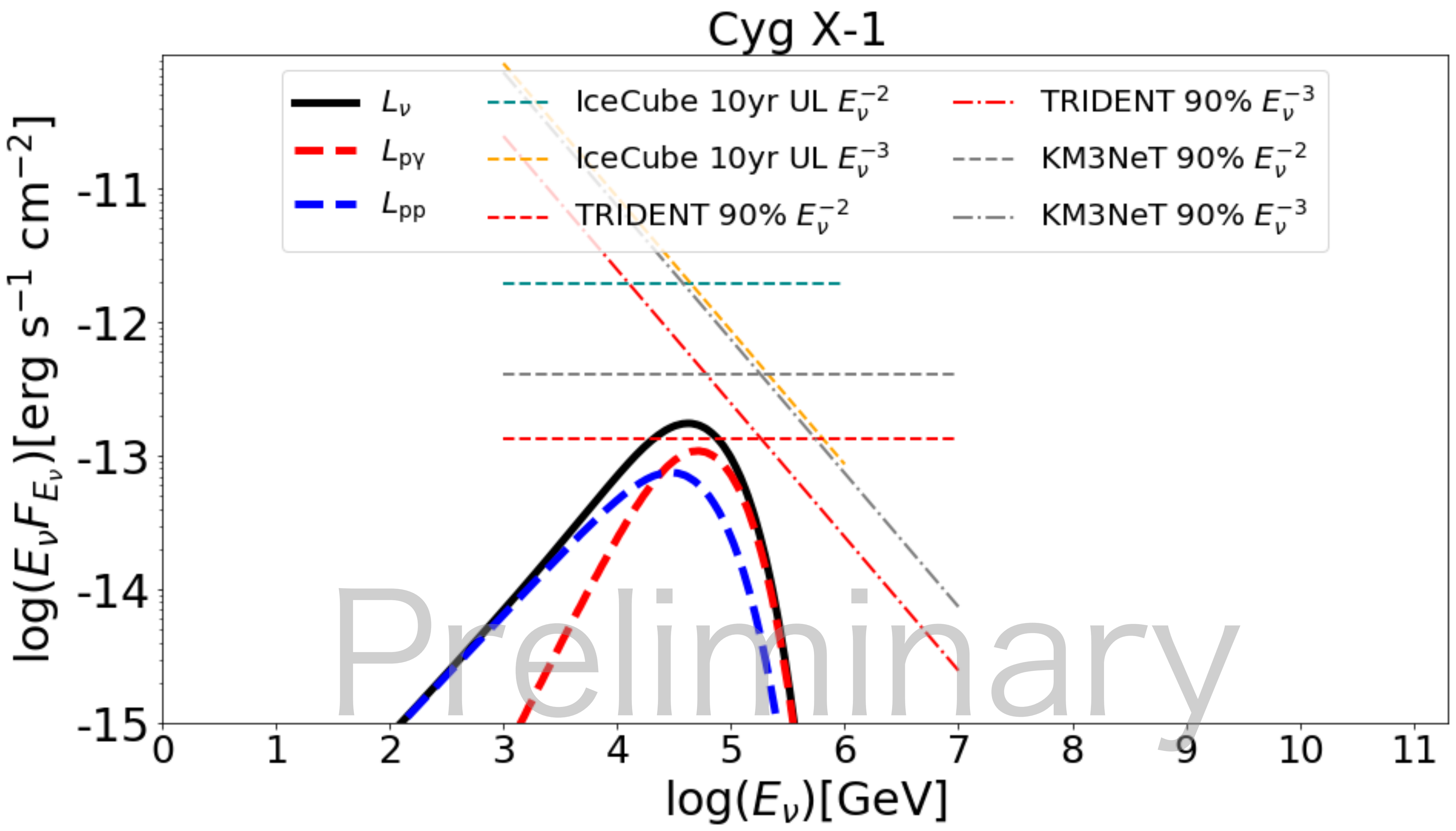


Thank you for listening!

Back Up Slide

Low conversion efficiency



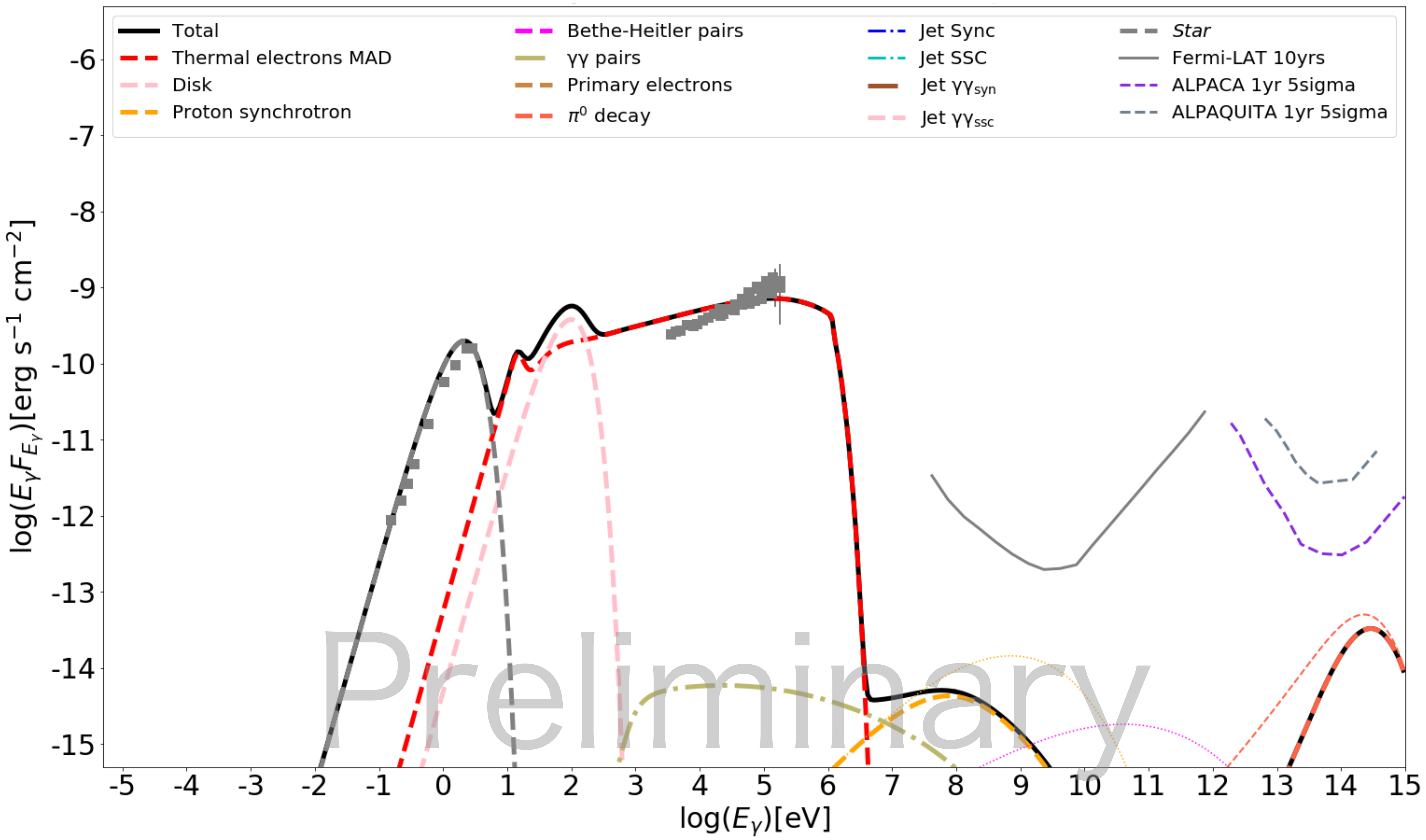


Due to the upper limits of **gamma-ray (LHAASO, HAWC, MAGIC)** and **neutrino (IceCube)**, we need to constrain the energy fraction of the nonthermal particle production to dissipation to be low.

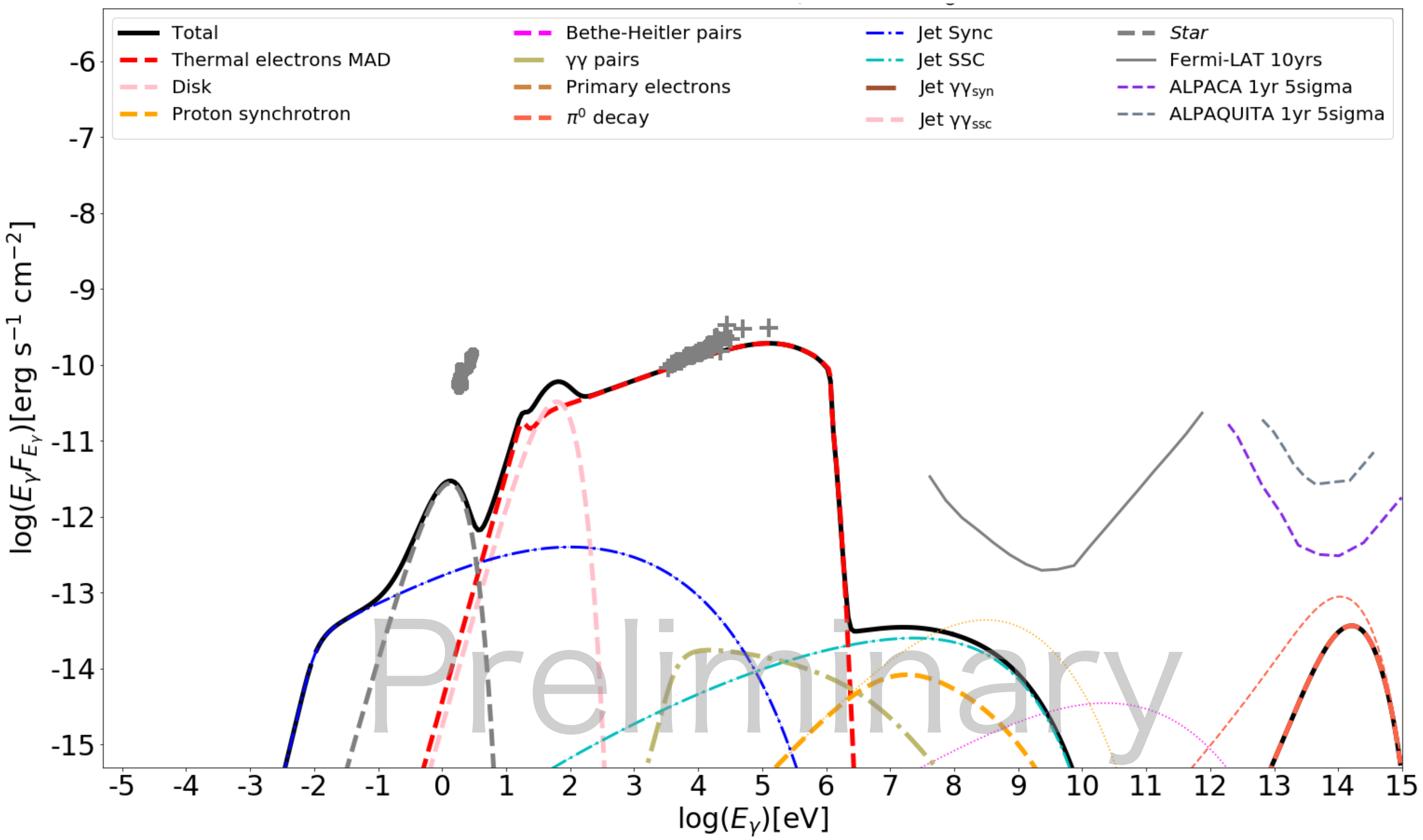
-> Acceleration efficiency of the turbulence will not be high in the BHXBs.

Applying our model to other BHXBs

GRO J1655-40



GX 339-4

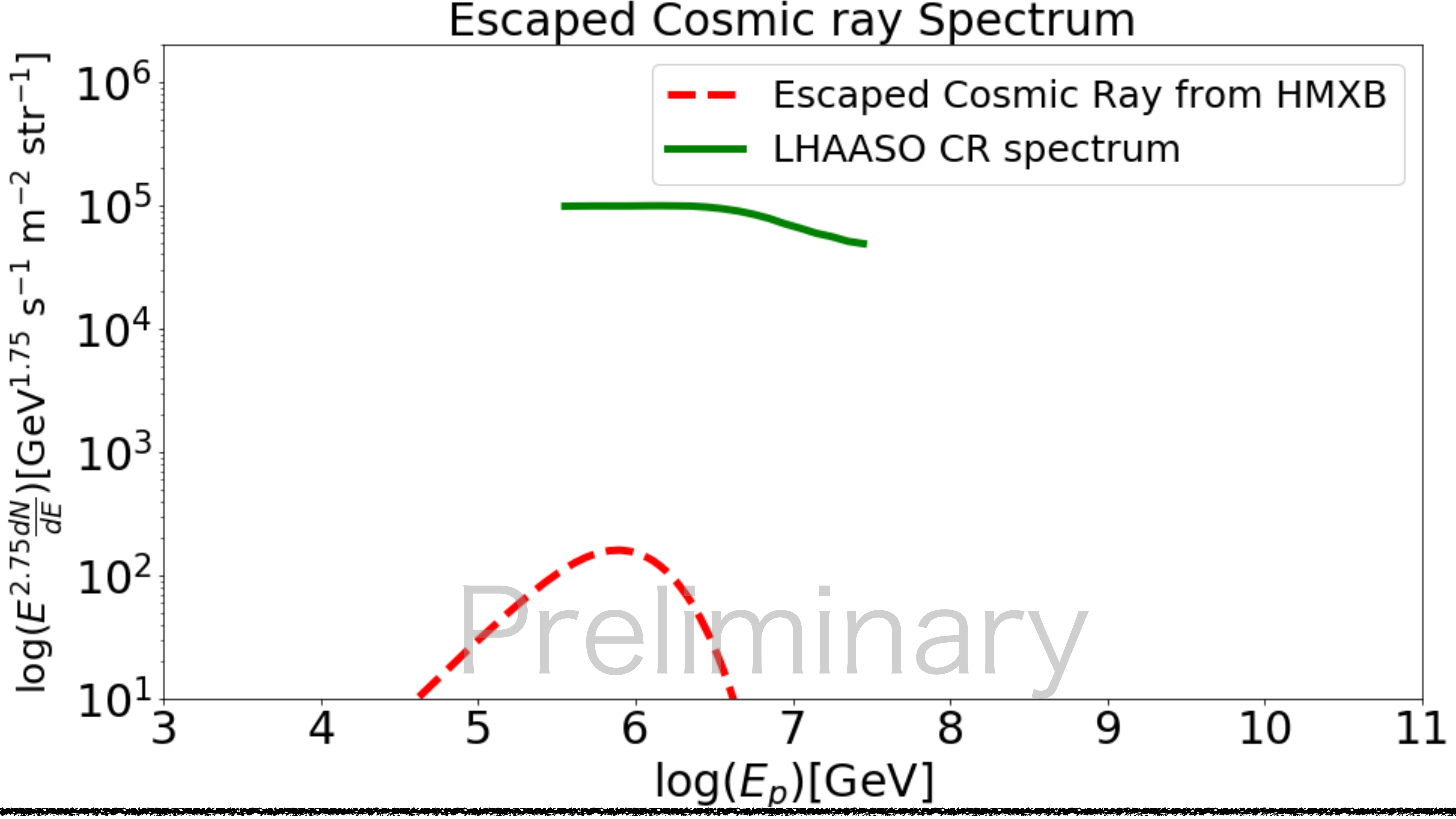


Our model explains the multi-wavelength data well.

-> The origin of the corona may be the MAD,

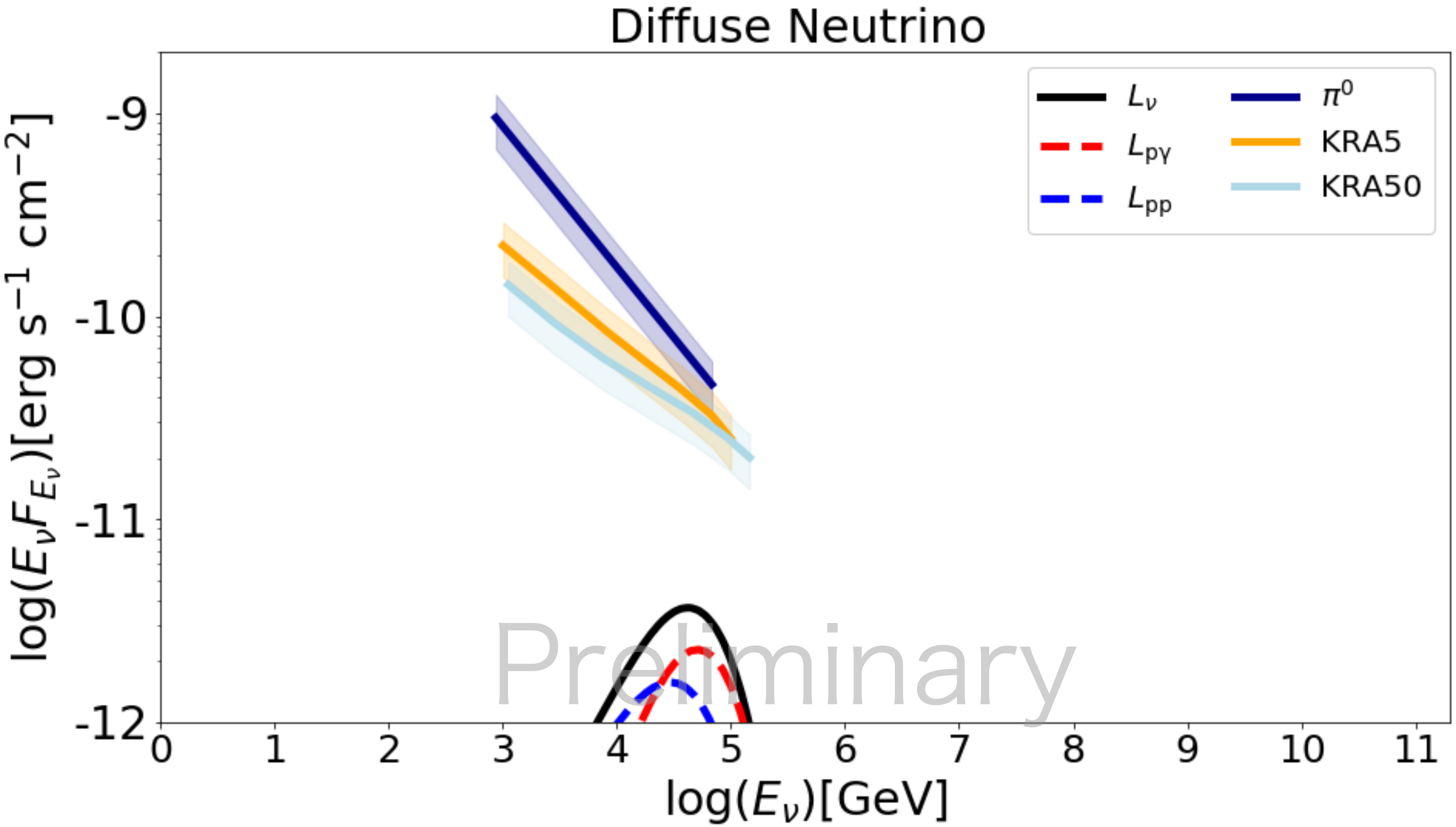
and Cygnus X-1 may be the special object.

Contribution to the CR spectrum



We calculate the escaped CRs from MADs and **estimate the contribution to the CR spectrum** assuming the number of BHXBs. We find that the **BHXBs will not be the source of the PeV CRs.**

Contribution to the galactic neutrino



We also estimate the contribution to the galactic neutrino.

Due to the **low duty cycle of the low/hard state (1.4%),**
BHXBs only contribute to a few 10 % of the galactic neutrino.

The estimate method of CR flux

We estimate the CR flux by equating the escape rate with the injection rate:

$$\underbrace{E\frac{dL_{\rm cr}}{dE}t_{\rm esc}}_{\text{Injection}} = \underbrace{E\frac{dU_{\rm cr}}{dE}V_{\rm halo}}_{\text{Escape}}.$$

$$\therefore E\frac{dL_{\rm cr}}{dE} = E\frac{dU_{\rm cr}}{dE}\frac{V_{\rm halo}}{t_{\rm esc}}$$

$$= E\frac{dU_{\rm cr}}{dE}\frac{M_{\rm gas}}{X_{\rm esc}}v$$

$$= E^2\Phi_{\rm cr}(E)4\pi\frac{M_{\rm gas}}{X_{\rm esc}}$$

$$\therefore E^2\Phi_{\rm cr}(E) = E\frac{dL_{\rm cr}}{dE}\frac{X_{\rm esc}}{4\pi M_{\rm gas}} \quad {\rm GeV\ s^{-1}\ cm^{-2}\ str^{-1}}\ ,$$

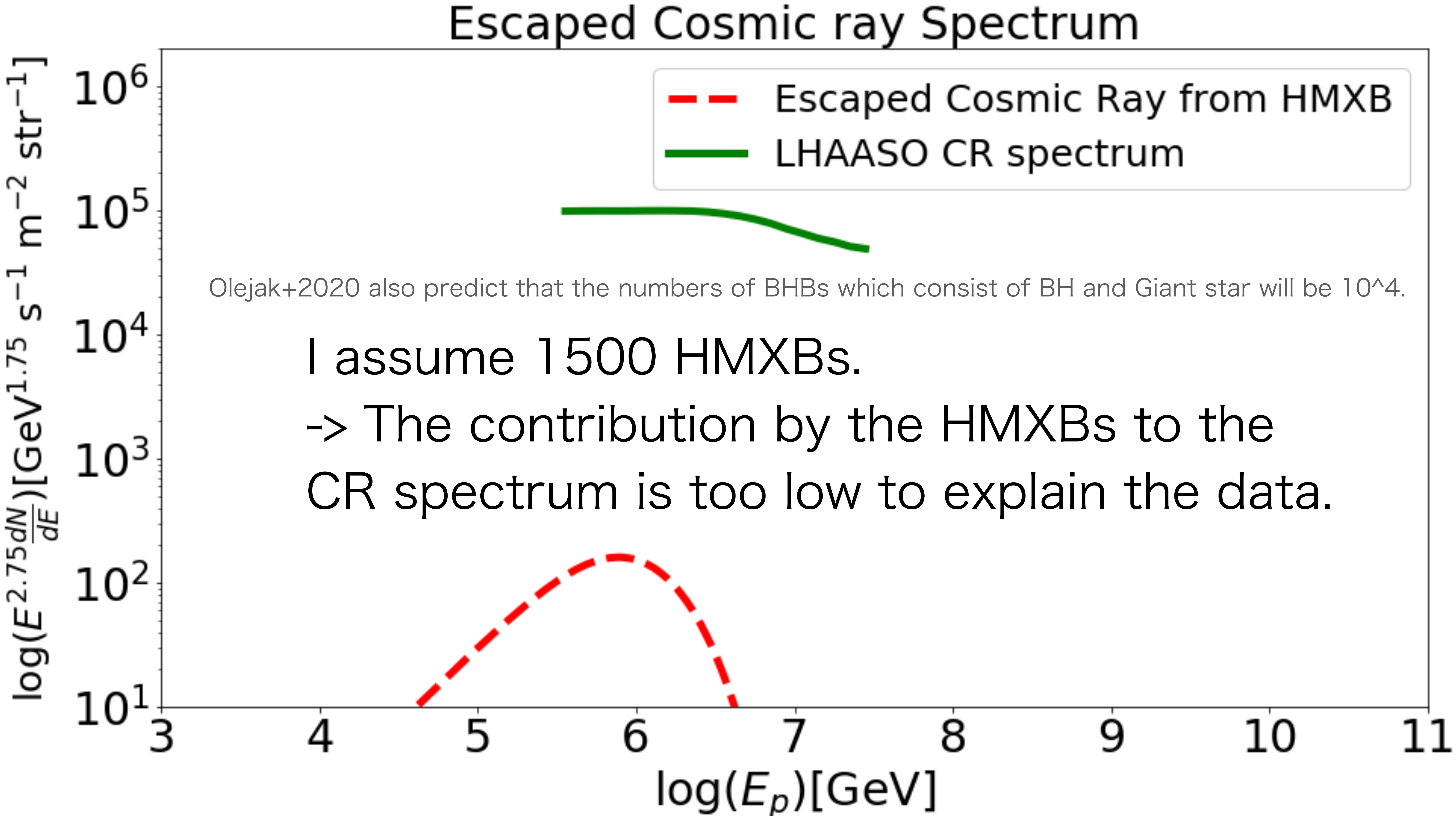
where $X_{\rm esc} = 8.7\left(\frac{E[{\rm GeV}]\ /e}{10\ {\rm GV}}\right)^{-\delta}$, $\delta = 0.33$, $M_{\rm gas} \simeq 10^{10}M_\odot$

e*V = eV
-> Volt = eV/e
e = 5x10^-10 esu
-> Volt ~= 2x10^9 eV/esu

$$250\ {\rm GV} = \frac{250\ {\rm GeV}}{\rm e}$$

Population synthesis of HMXB

BlackCAT(Corral+2016) -> 1280 HMXBs in our galaxy (by ORP).
Population synthesis(Shao+2020) -> ~1500 HMXBs in our galaxy.



Olejak+2020 also predict that the numbers of BHBs which consist of BH and Giant star will be 10^4.

I assume 1500 HMXBs.
-> The contribution by the HMXBs to the CR spectrum is too low to explain the data.

Timescales of the MAD model

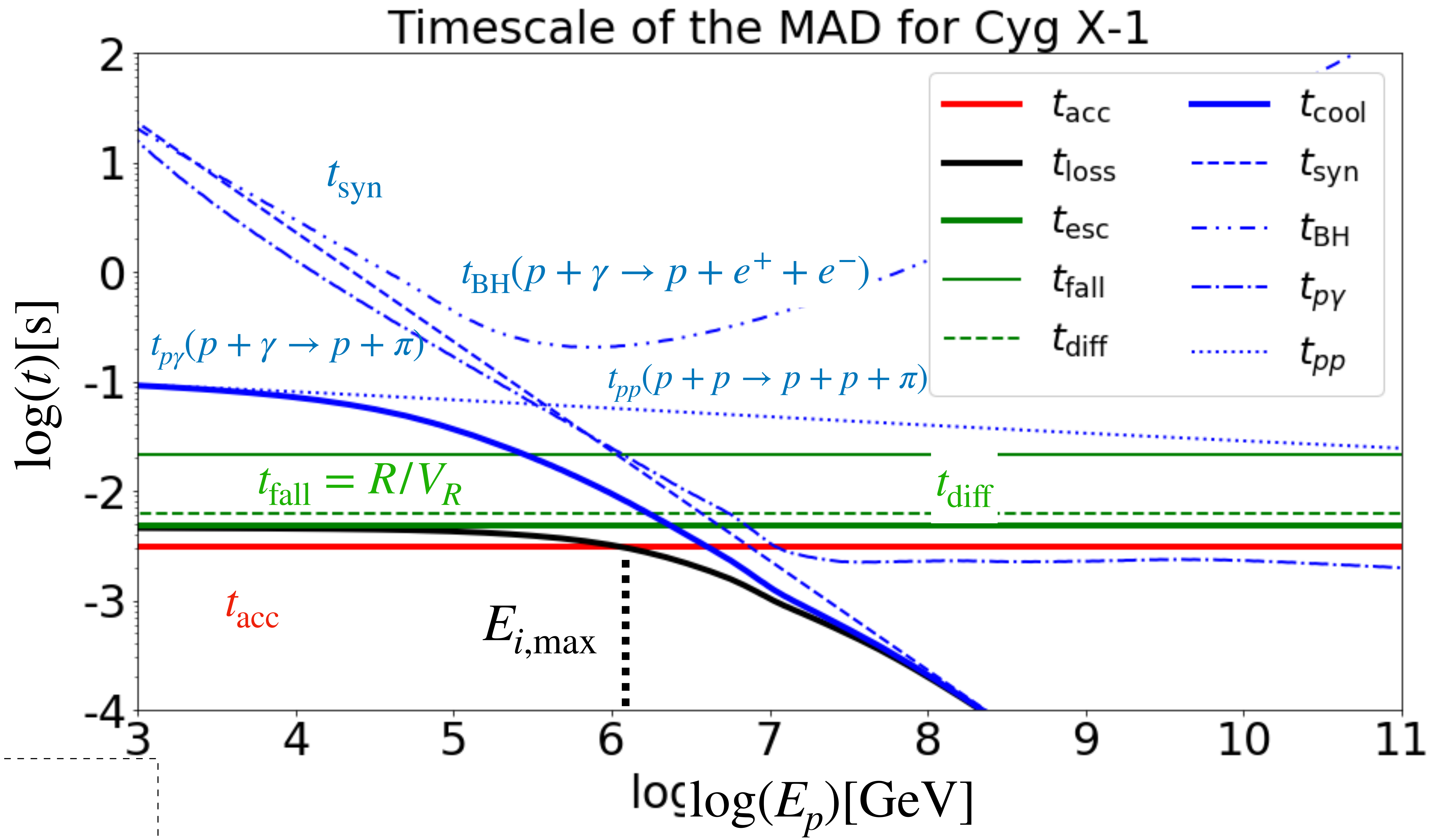
We assume $q = 2$

$$D_E \propto \beta_A^2 E^2 \quad (P_k \propto k^{-2})$$

$$t_{\rm acc} = \frac{p^2}{D_{pp}} \propto m\beta$$

$$t_{\rm diff} = \frac{R^2}{D_{zz}} \propto m$$

$$t_{\rm syn} = \frac{E}{P_{\rm syn}} \propto E^{-1} m \dot{m}^{-1}$$



$$\boxed{E_{i,\max} : t_{\rm acc} = t_{\rm loss}}$$

Injection Term

$$\dot{N}_{E_{i,\rm inj}} = \dot{N}_0 \left(\frac{E_i}{E_{i,\max}}\right)^{-s_{\rm inj}} \exp\left(-\frac{E_i}{E_{i,\max}}\right)$$

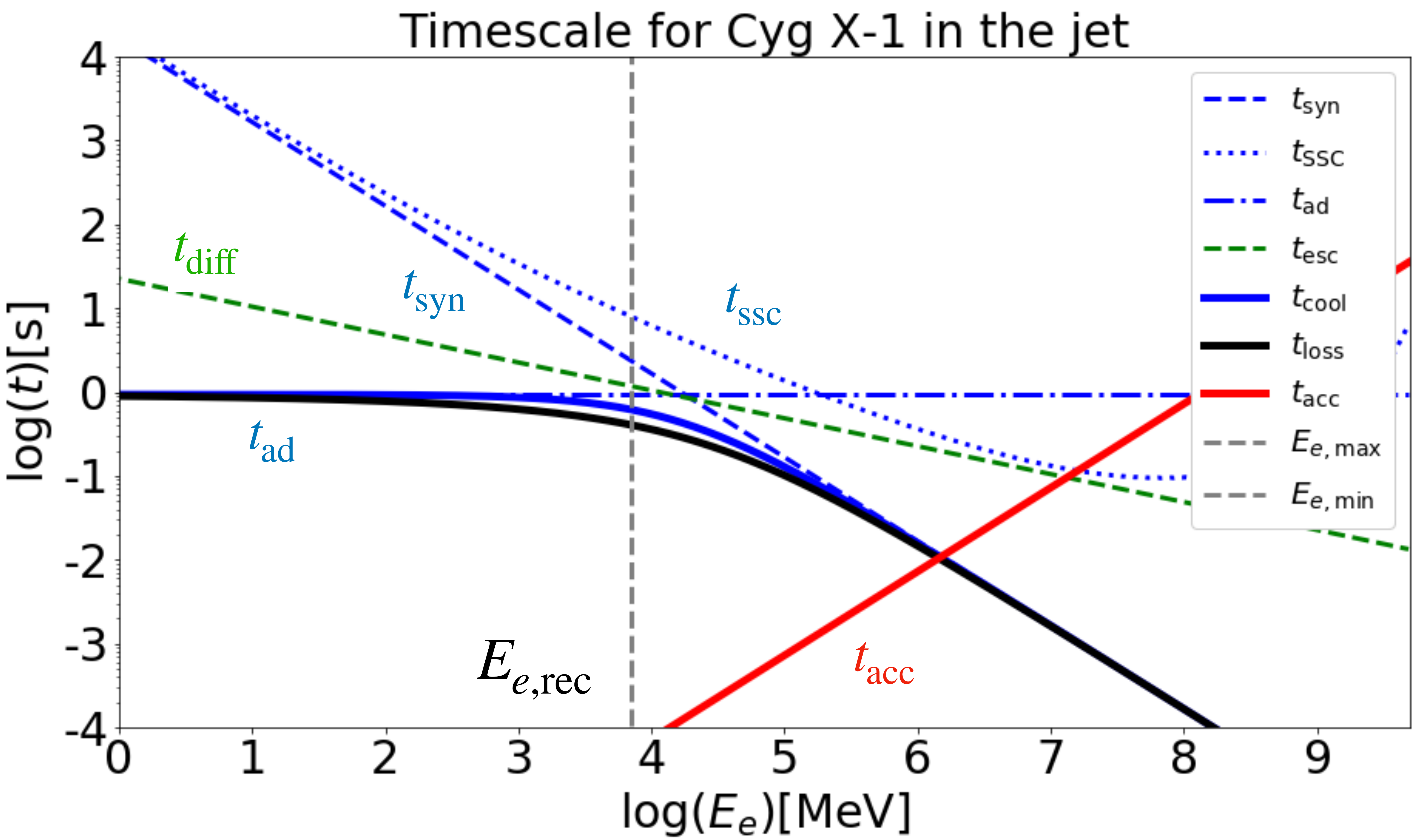
Timescales inside the jet

We assume $q = 5/3$

(Kolmogorov spectrum)

$D_{zz} \propto E^{-1/3} \quad (P_k \propto k^{-5/3})$

$$\left\{ \begin{array}{l} t_{\rm acc} = \dfrac{\gamma m_e c}{eB\beta_{\rm rec}} \propto E \\[2ex] t_{\rm diff} = \dfrac{R_b^2}{D_{zz}} \propto E^{-1/3} \\[2ex] t_{\rm syn} = \dfrac{E}{P_{\rm syn}} \propto E^{-1} \\[2ex] t_{\rm ad} = 3\dfrac{R_b}{c} \propto E^{0} \end{array} \right.$$



$$\boxed{E_{e,\max} = \min\left[\left(\frac{\delta B}{B}\right)^2 \xi\sigma m_e c^2, t_{\rm acc} = t_{\rm loss}\right]}$$

Cygnus X-1

Physical quantities

$B = 1.4 \times 10^7\ \mathrm{G}$

$n_p = 2.1 \times 10^{16}\ \mathrm{cm}^{-3}$

$\theta_e = 0.24$

$T_{\mathrm{disk}} = 21\ \mathrm{eV}$

MAD parameter

$M = 21 M_\odot$

$\dot{m} = 0.1$

$R = 10 R_g$

$\alpha = 0.3$

$\beta = 0.1$

$\epsilon_{\mathrm{dis}} = 0.1$

$\epsilon_{\mathrm{NT}} = 0.0003$

$\eta_{\mathrm{diff}} = 3.0 \rightarrow l_{\mathrm{mfp}} = \frac{1}{3}\eta_{\mathrm{diff}} r_L$

$s_{\mathrm{inj}} = 1.16$

$R_{\mathrm{disk}} = 250 R_g$

Physical quantities

$B_{\mathrm{jet}} = 5.96 \times 10^2\ \mathrm{G}$

$n_p = 1.9 \times 10^{10}\ \mathrm{cm}^{-3}$

$\sigma_\pm = 1.12 \times 10^2$

$L_j = 1.55 \times 10^{38}\ \mathrm{erg\ s^{-1}}$

$L_e = 9.7 \times 10^{34}\ \mathrm{erg\ s^{-1}}$

$\Gamma = 2.55,\ \beta = 0.92$

$\theta_j = 27.5\ \ \mathrm{deg}$

Jet parameter

$\sigma_{\mathrm{ent}} = 0.001$

$R_j = 10^3 R_g$

$p = 2.1$

$\xi = 10^3$

$\delta B / B = 0.35$

Companion Star

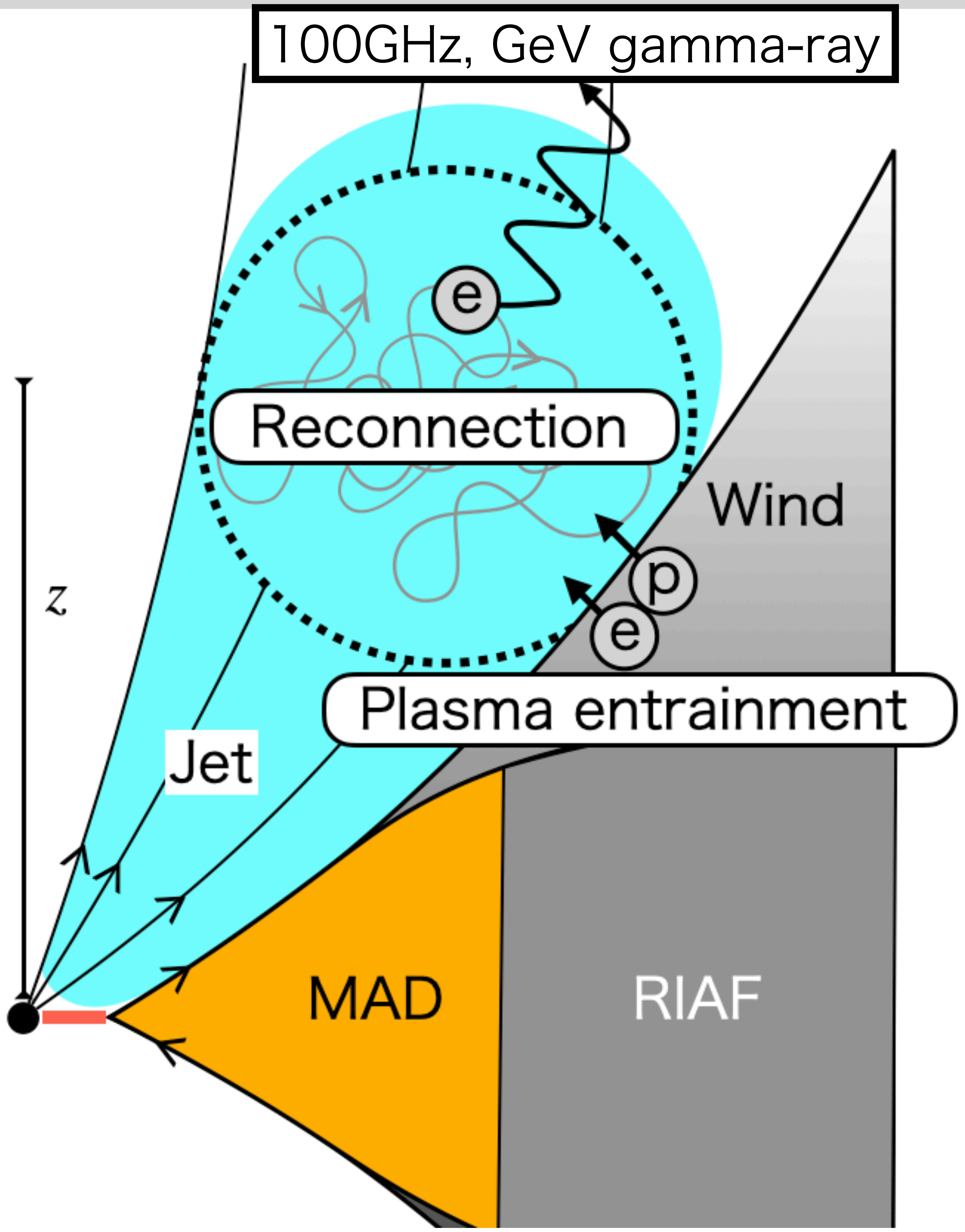
$T_{\mathrm{star}} = 3.1 \times 10^4\ \mathrm{K}$

$L_{\mathrm{star}} = 1.6 \times 10^{39}\ \mathrm{erg\ s^{-1}}$

$a_{\mathrm{binary}} = 3.7 \times 10^{12}\ \mathrm{cm}$

Kantzas+2021

Emission from the Jet



We estimate the physical quantities via conservation of energy

$$B \simeq 1.7 \times 10^4 \dot{m}_{-2}^{-1/2} m_1^{-1/2} R_{j,3}^{-1} \left(\frac{\sigma_{\rm dis}/(1+0.5\sigma_{\rm dis})}{0.66} \right)^{1/2} \ {\rm G}$$

$$n_e \simeq 2.4 \times 10^{10} \dot{m}_{-2} m_1^{-1} \sigma_{\rm ent}^{-1} R_{j,3}^{-2} \ {\rm cm}^{-3}$$

$$L_j \simeq 7.4 \times 10^{36} \dot{m}_{-2} m_1 \ {\rm erg\ s}^{-1}$$

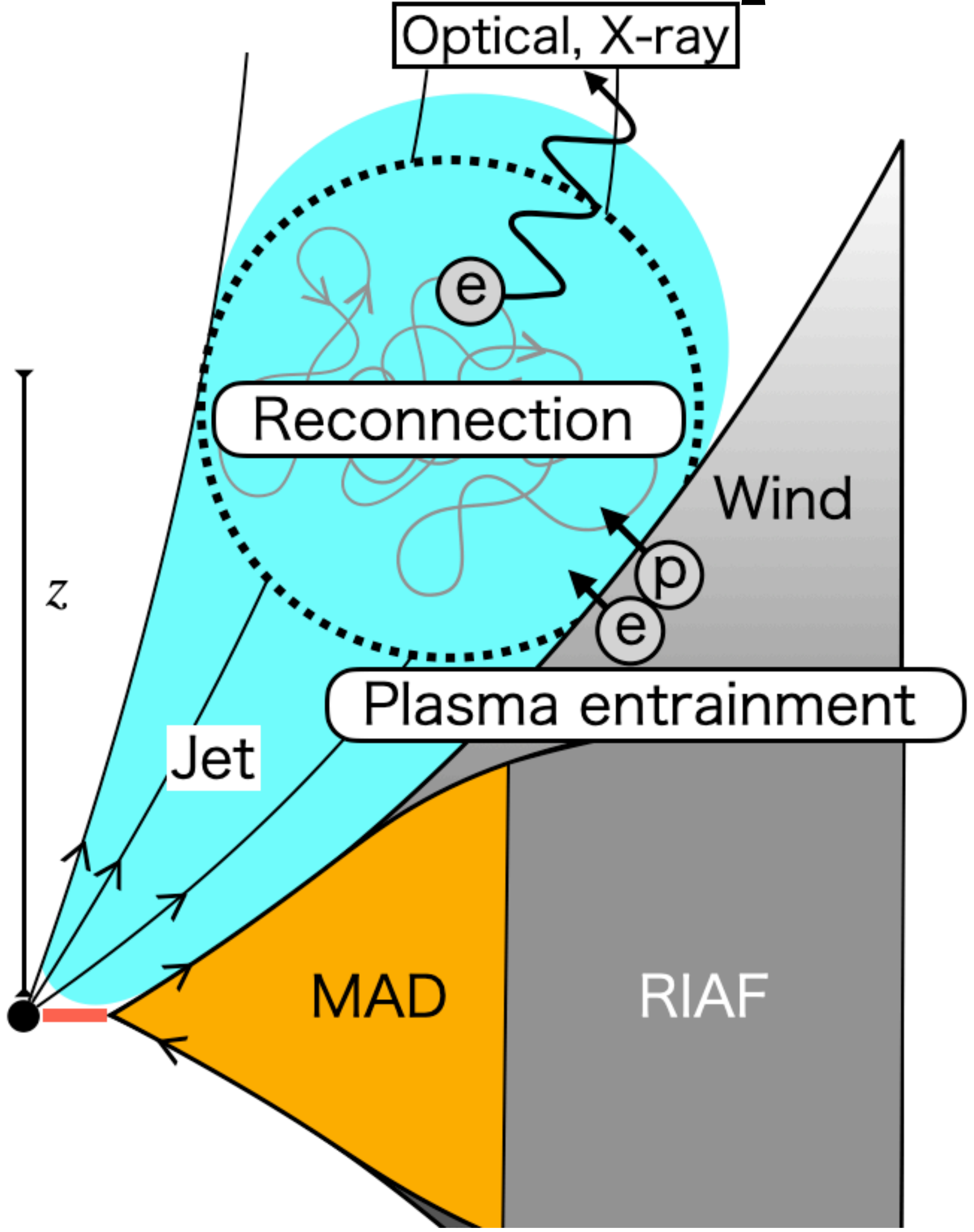
Magnetic reconnection inside the jet accelerates the particles.

- Steady & one-zone approximations
- Nonthermal Electrons

Transport Equation

$$\underbrace{-\frac{d}{dE_e}\left(\frac{N_{E_e}E_e}{t_{e,\rm cool}}\right)}_{\text{Cooling}} = \underbrace{\dot{N}_{E_e,\rm inj}}_{\text{Injection}} - \underbrace{\frac{N_{E_e}}{t_{\rm esc}}}_{\text{Escape}}$$

Basic equation of Jet-MAD model



Entrainment ($\sigma = \sigma_{\rm ent}$)

$$L_j \approx \left(1 + 0.5\sigma_{\rm ent}\right) n_p m_p c^2 \pi R_b^2 c$$

$$L_j \approx \eta \dot{M} c^2 = \eta \dot{m} L_{\rm Edd}$$

→

Number density

$$n_p = \frac{L_j}{1 + 0.5\sigma_{\rm ent}} \frac{1}{m_p c^2} \frac{1}{\pi R_b^2 c}$$

After dissipation ($\sigma = \sigma_{\rm dis}$)

$$L_j = \left(u_e + u_p + \frac{B^2}{8\pi}\right) \pi R_b^2 c$$

$$\sigma_{\rm dis} = \frac{B^2}{4\pi(u_e + u_p)}$$

→

Magnetic Field

$$B = \sqrt{4\pi u_e \sigma_{\rm dis}}$$

$$u_e + u_p = \frac{L_j}{1 + 0.5\sigma_{\rm dis}} \frac{1}{\pi R_b^2 c}$$

Magnetic reconnection inside the blob accelerates the particles.

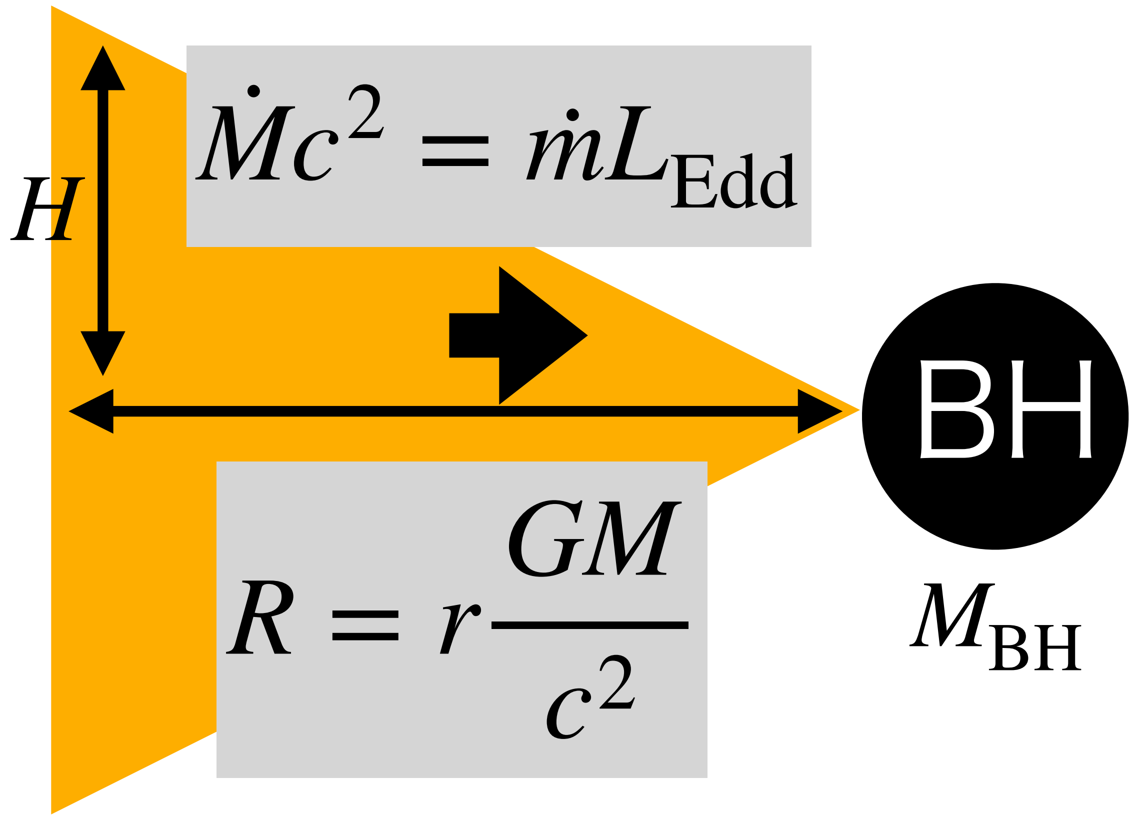
- Steady & one-zone approximations
- Nonthermal Electrons

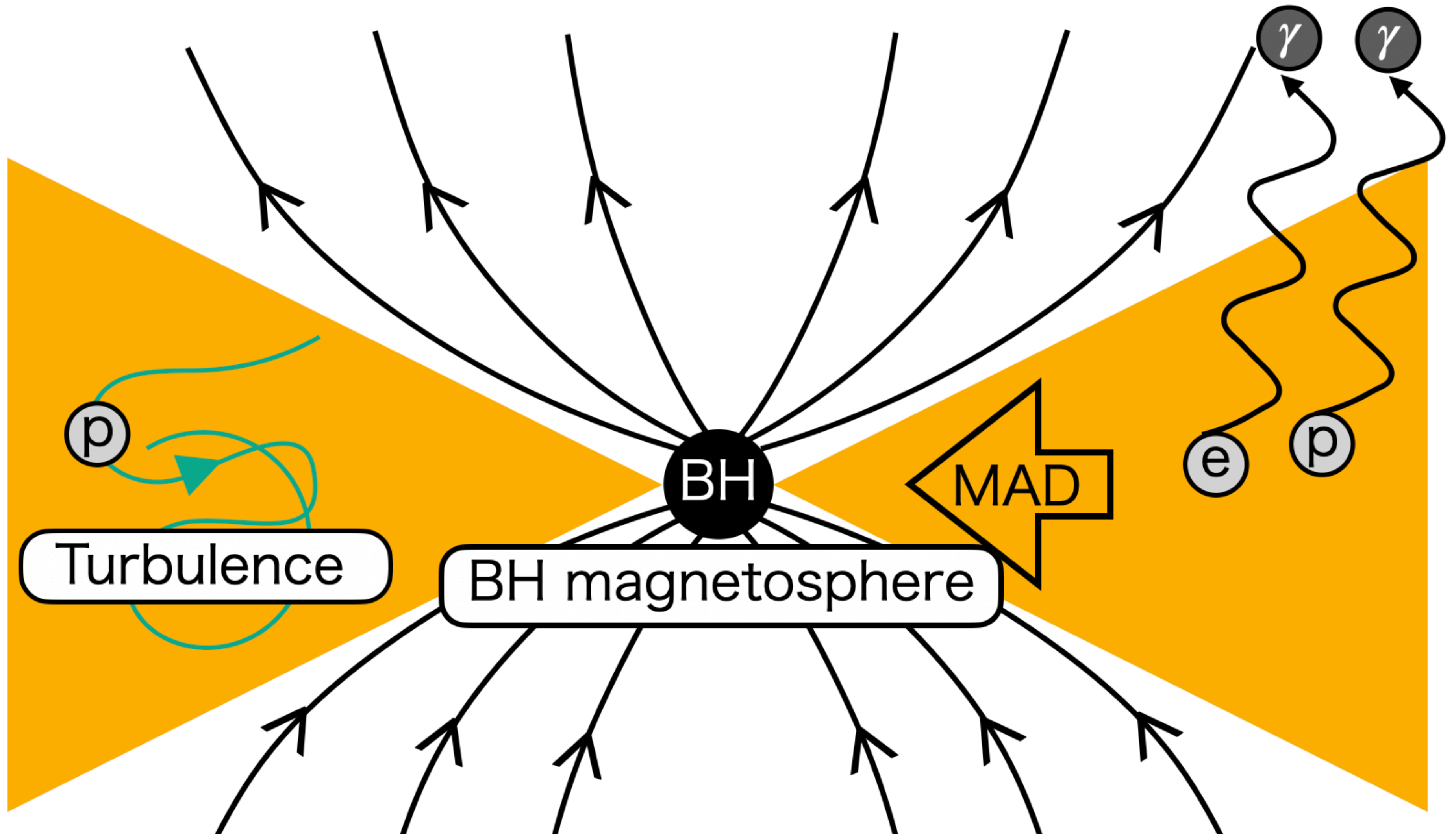
Transport Equation

$$\underbrace{-\frac{d}{dE_e}\left(\frac{N_{E_e} E_e}{t_{e,\rm cool}}\right)}_{\text{Cooling}} = \underbrace{\dot{N}_{E_e,\rm inj}}_{\text{Injection}} - \underbrace{\frac{N_{E_e}}{t_{\rm esc}}}_{\text{Escape}}$$

Parameters of the MAD model

Steady & one-zone approximations





9 Parameters

$M_{\rm BH} = \underline{m}M_\odot$ $\dot{M}c^2 = \boxed{\dot{m}}L_{\rm Edd}$

$R = \underline{r}\frac{GM}{c^2}$ Viscosity: $\underline{\alpha}$

$\underline{\beta} = \frac{P_{\rm gas}}{P_{\rm B}}$ $L_{\rm tot} = \underline{\epsilon_{\rm dis}}\dot{M}c^2$

$L_{\rm non,thml} = \underline{\epsilon_{\rm NT}}\epsilon_{\rm dis}\dot{M}c^2$

$l_{\rm mfp} = \underline{\eta}\frac{E_i}{eB}$ Injection index: $\underline{s_{\rm inj}}$

Each parameter: Restricted physically.
-> The MAD model cannot explain any photon spectrum

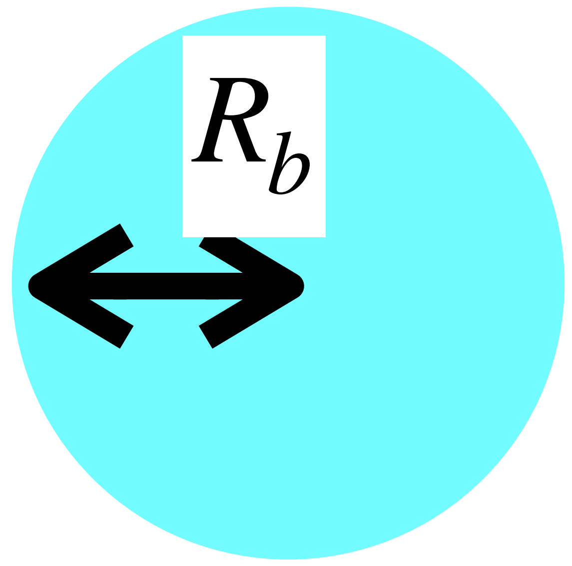
Cyg X-1: $n_p \simeq 2.1 \times 10^{16}\ {\rm cm}^{-3}$, $B = 1.4 \times 10^7$ G

Mass density : $\rho \approx \frac{\dot{M}}{4\pi RHV_R} \propto \dot{m}m^{-1}r^{-3/2}\alpha^{-1}$

Magnetic field : $B = \sqrt{\frac{8\pi\rho C_s^2}{\beta}} \propto \dot{m}^{1/2}m^{-1/2}r^{-5/4}\alpha^{-1/2}\beta^{-1/2}$

Parameters inside the jet

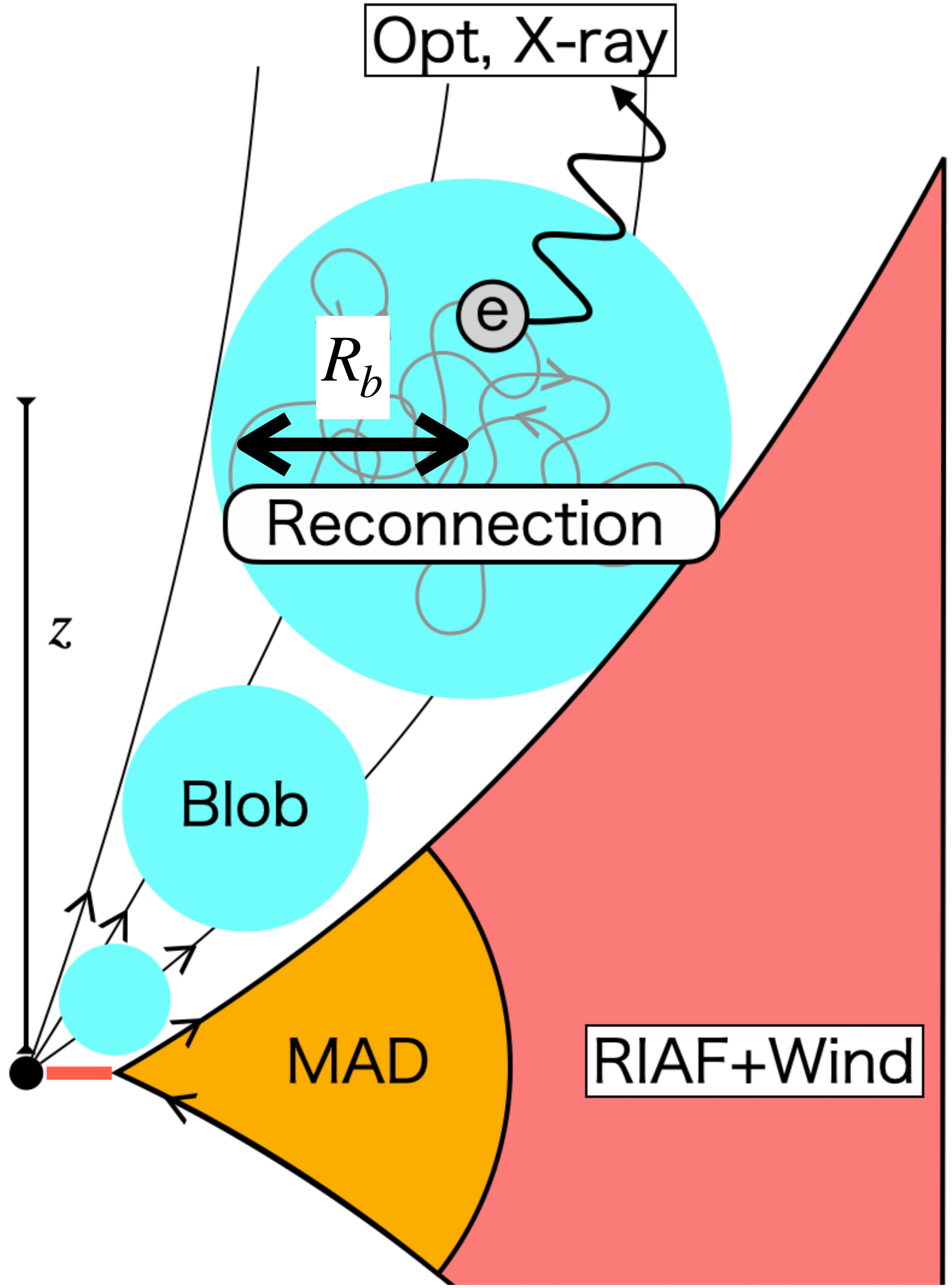
Steady & one-zone approximations



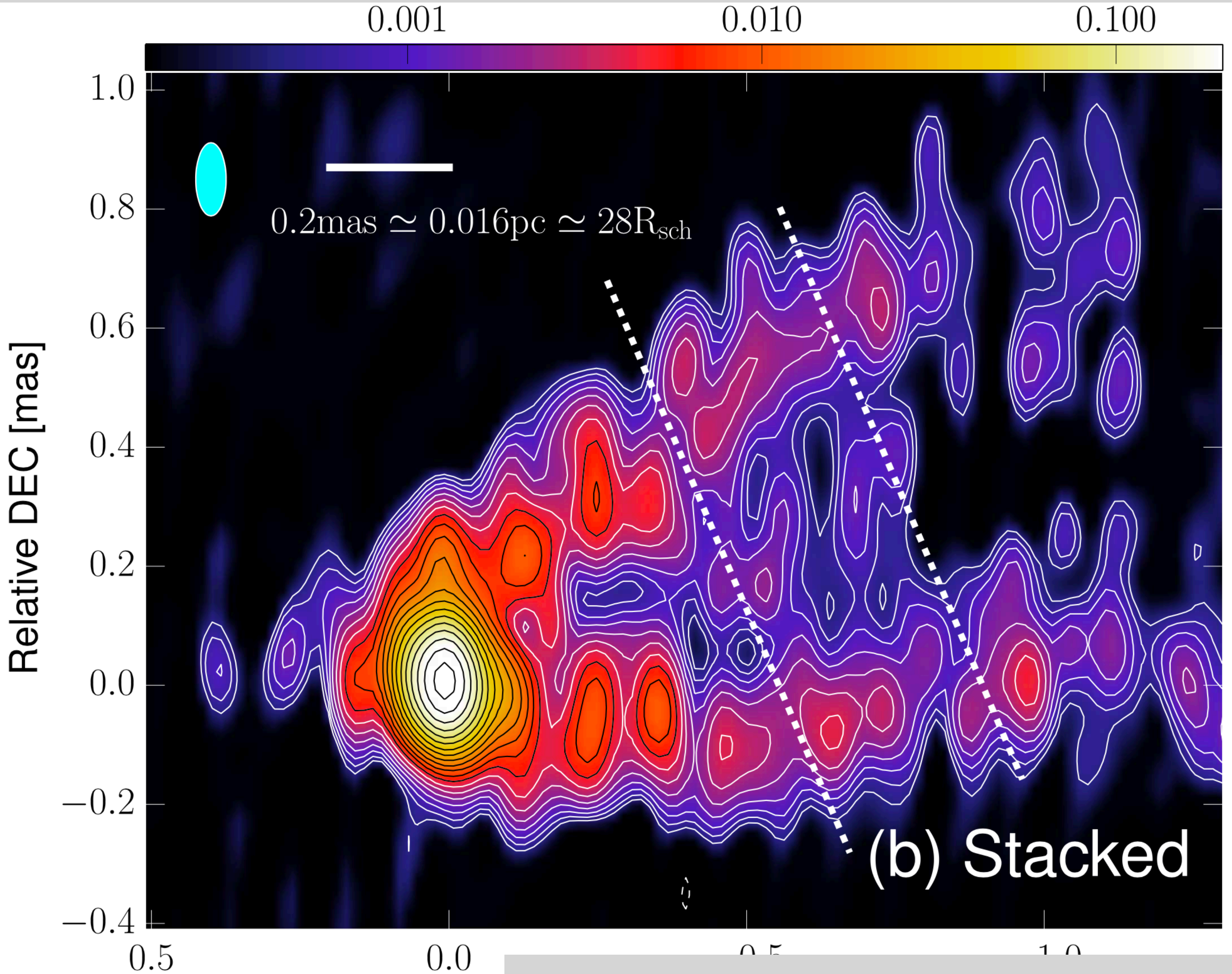
Blob size = $R_b \times R_g$

5 Parameters

$\sigma_{\rm ent}$	p	ξ	$\delta B/B$	$R_b[R_g]$
0.001	2.1	10^3	0.33	10^3



Scenario of Jet-MAD model



Kim+2018

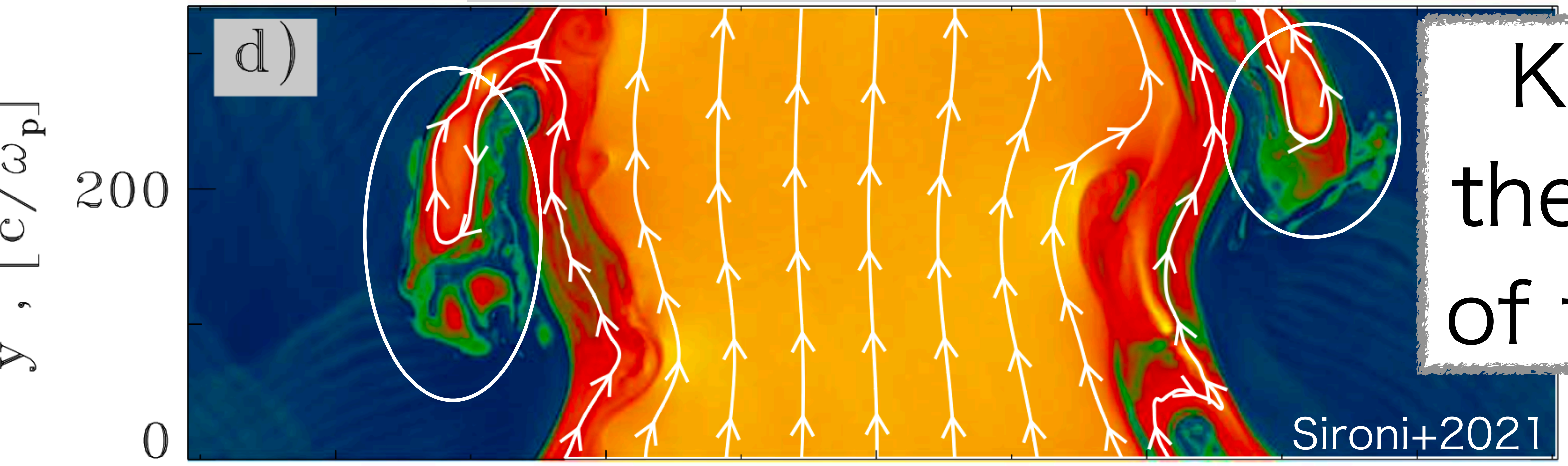
Limb brightening

▼

Dissipation at the edge of the jet

2D-PIC Simulation

KHI: Kelvin-Helmholtz instability

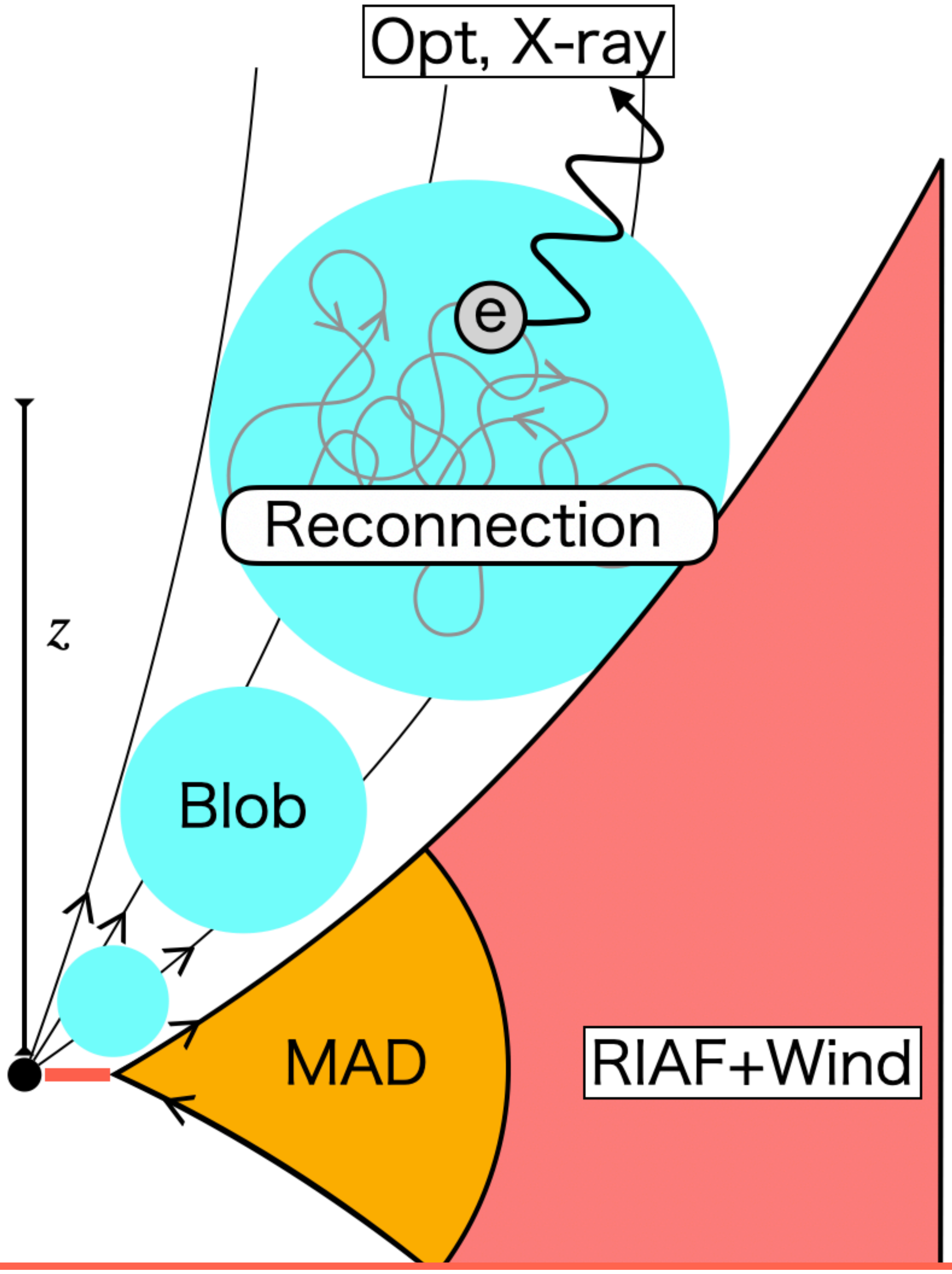


Sironi+2021

KHI at the edge of the jet

We consider that the KHI induces **turbulence** inside the jet, and this turbulence induces **magnetic reconnection.**

Schematic image of Jet-MAD model



Magnetic reconnection inside the blob accelerates the particles.

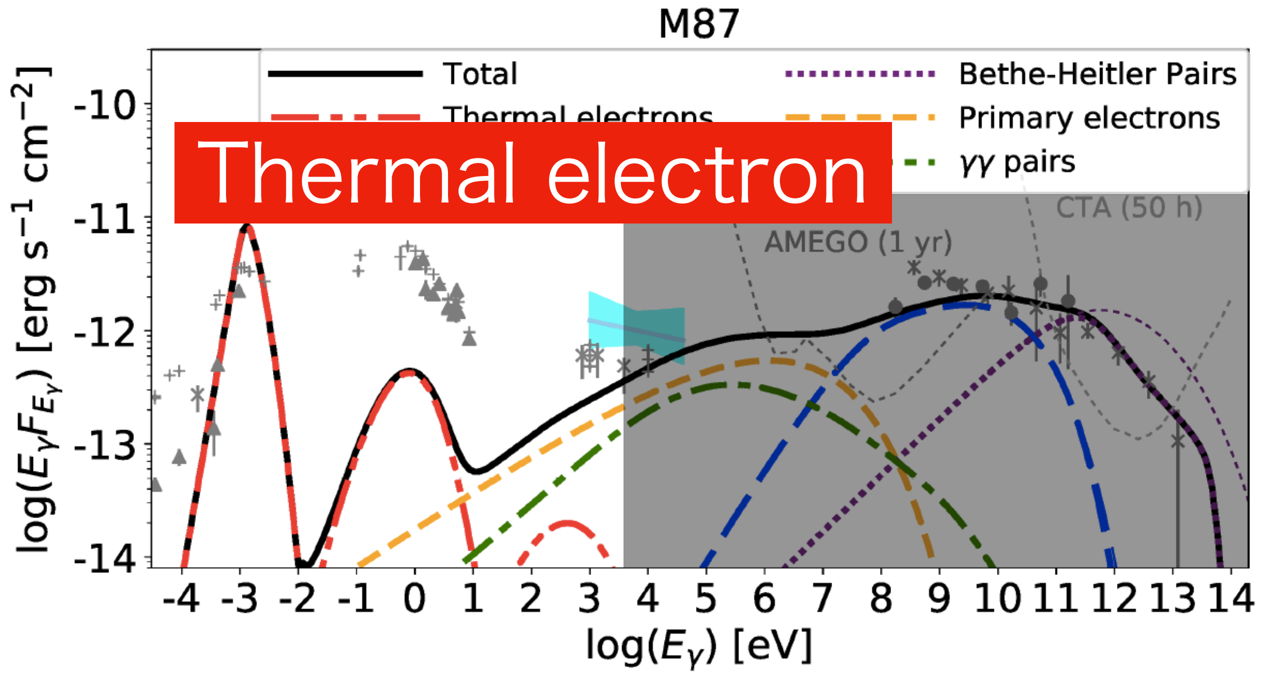
Calculation Method (Thermal)

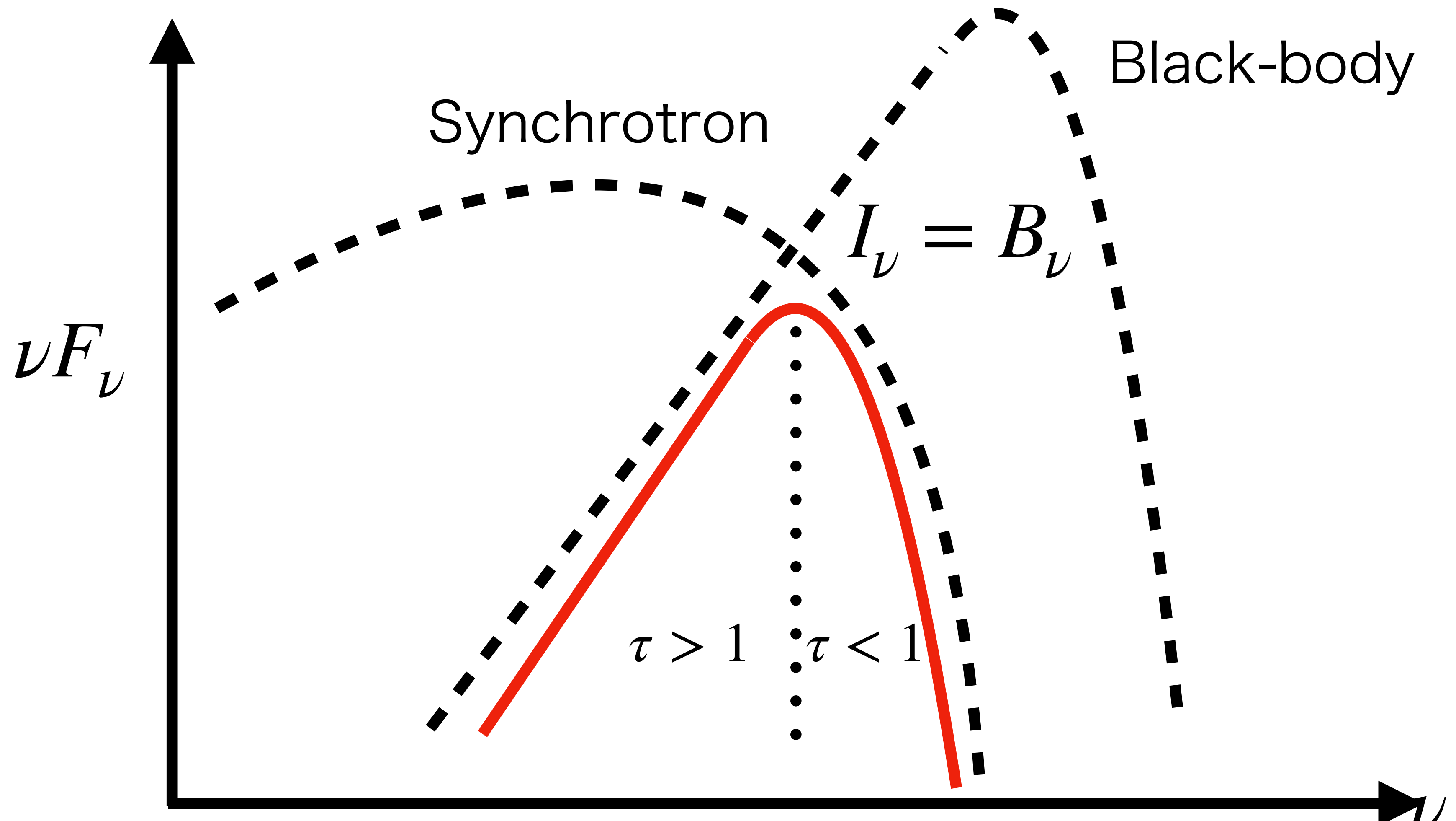
Parameter set: $M_{\rm BH} = mM_\odot,\ \dot{M}c^2 = \dot{m}L_{\rm Edd},\ R = r\dfrac{GM}{c^2},\ \alpha,\ \beta,\ \epsilon_{\rm NT},\ \epsilon_{\rm dis},\ \eta,\ s_{\rm inj}$

$L_{\rm tot} = \epsilon_{\rm dis}\dot{M}c^2$

$L_{\rm non,thml} = \epsilon_{\rm NT}\epsilon_{\rm dis}\dot{M}c^2$

Thermal electron component





$\tau > 1$: Optically thick by the synchrotron-self absorption and Black-body radiation.

$\tau < 1$: Synchrotron radiation by the thermal electron

- Radiation process

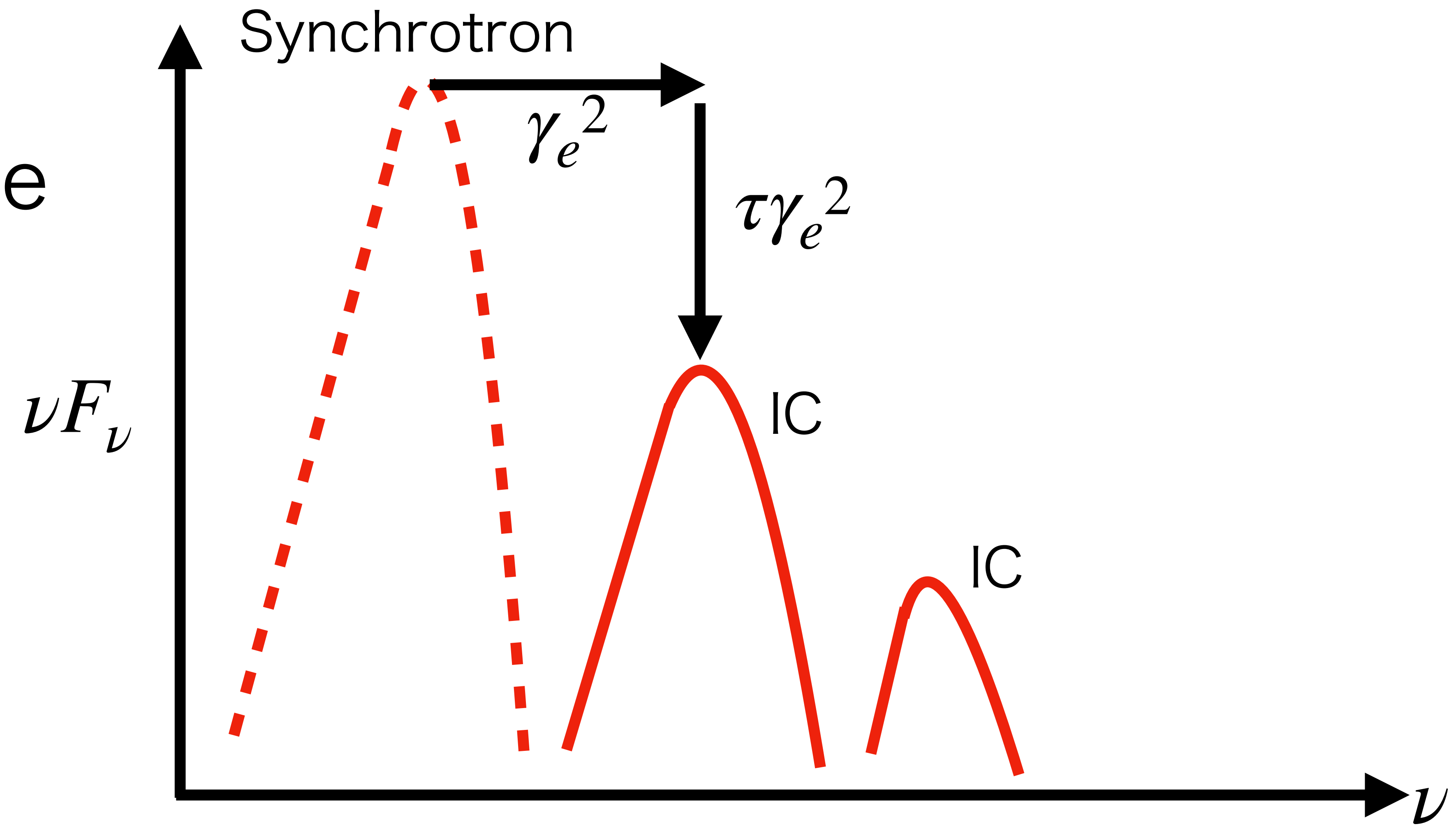
  Synchrotron + Inverse Compton (IC) (+Bremsstrahlung)

- Electron heating rate: $\dfrac{\mathcal{Q}_e}{\mathcal{Q}_p} \approx \dfrac{P_{\rm compr}}{P_{AW}} \simeq 1$ Kawazura+2020

- By solving $L_{\nu,{\rm thml}} \approx \left(\dfrac{\mathcal{Q}_e}{\mathcal{Q}_p}\right)\left(1-\epsilon_{\rm NT}\right)\epsilon_{\rm dis}\dot{m}L_{\rm Edd}$ for the temperature

-> Calculate photon spectra

Calculate iteratively

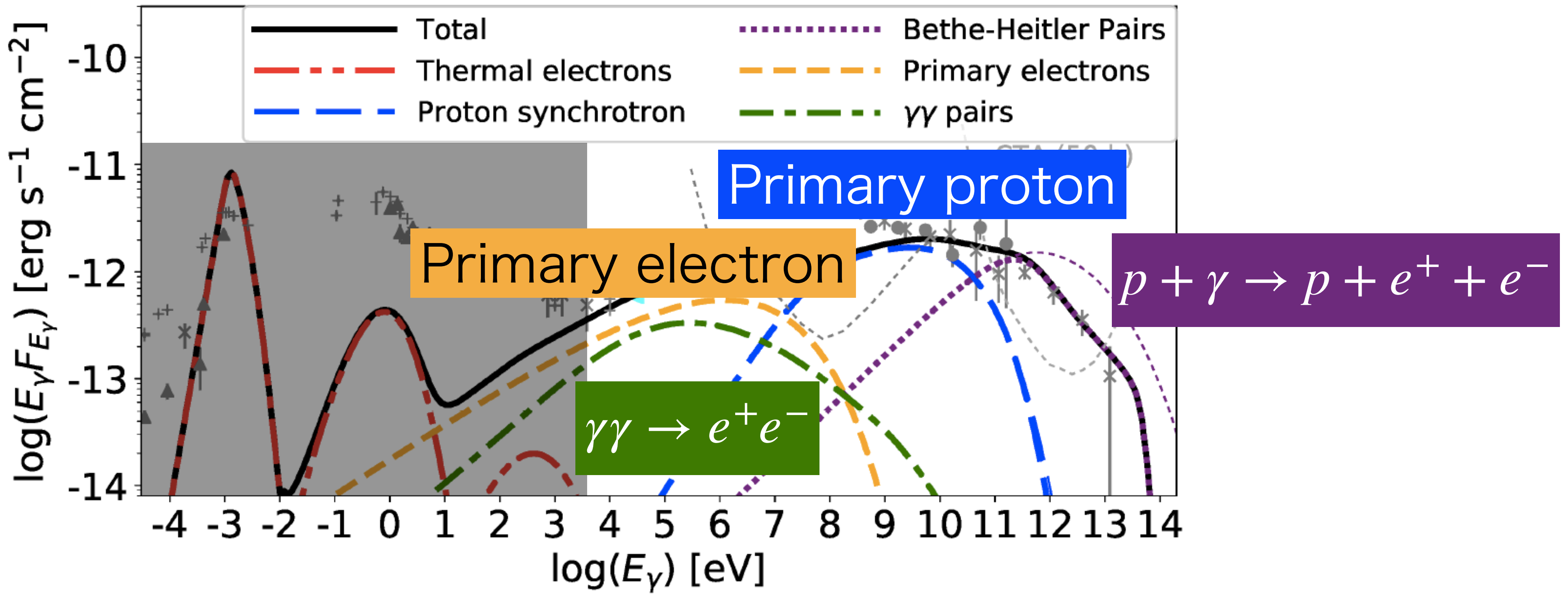


Calculation Method (Non-thermal)

$Q_{\rm tot} = \epsilon_{\rm dis}\dot{M}c^2$

Parametr set: $M_{\rm BH} \underset{\rm M87}{=} mM_\odot,\ \dot{M}c^2 = \dot{m}L_{\rm Edd},\ R = r\frac{GM}{c^2},\ \alpha,\ \beta,\ \epsilon_{\rm NT},\ \epsilon_{\rm dis},\ \eta,\ s_{\rm inj}$ $Q_{\rm non,thml} = \epsilon_{\rm NT}\epsilon_{\rm dis}\dot{M}c^2$

Non-thermal component



Kimura & Toma 2020

Transport eq. $$-\frac{d}{dE_i}\left(\frac{N_{E_i}E_i}{t_{i,\rm cool}}\right) = \dot{N}_{E_i,\rm inj} - \frac{N_{E_i}}{t_{\rm esc}}$$

$$\dot{N}_{E_{i,\rm inj}} \approx \dot{N}_0\left(\frac{E_i}{E_{i,\rm cut}}\right)^{-s_{\rm inj}} \exp\left(-\frac{E_i}{E_{i,\rm cut}}\right)$$

- $L_e \approx \int \dot{N}_{E_{e,\rm inj}} E_e dE_e \approx \left(\frac{\mathcal{Q}_e}{\mathcal{Q}_p}\right)\epsilon_{\rm NT}\epsilon_{\rm dis}\dot{M}c^2$ & Transport eq.
- $L_p \approx \int \dot{N}_{E_{p,\rm inj}} E_p dE_p \approx \epsilon_{\rm NT}\epsilon_{\rm dis}\dot{M}c^2$ & Transport eq.

① Calculate the number spectrum of the non-thermal particle
-> Calculate the photon spectrum.

- ② By using the number spectrum of non-thermal protons and the photon spectra, we calculate the number spectrum of secondary electron-positron pairs by the Bethe-Heitler process ($p+\gamma \to p+e^+ + e^-$). -> We calculate the photon spectrum.
- ③ By using the total photon spectra, we calculate the number spectrum of secondary electron-positron pairs by the two-photon interaction ($\gamma + \gamma \to e^+ + e^-$)
-> We calculate the photon spectrum.

Calculate iteratively

Injection index of the MAD model

Transport Equation in Energy space

Stawarz&Petrosian 2008

$$\frac{\partial n(p,t)}{\partial t} = \frac{\partial}{\partial p}\left[D(p)\frac{\partial n(p,t)}{\partial p}\right] - \frac{\partial}{\partial p}\left[\left(\frac{2D(p)}{p} + \langle\dot{p}\rangle\right)n(p,t)\right] - \frac{n(p,t)}{t_{\rm esc}} + \tilde{Q}(p,t).$$

Diffusion in energy Cooling Escape Injection

- $D_{pp} \propto p^{q}, \;\; D_{zz} \propto p^{2-q} \rightarrow t_{\rm acc} \propto p^{2-q}, t_{\rm esc} \propto p^{q-2}$
- steady-state -> $\frac{\partial n}{\partial t} = 0$

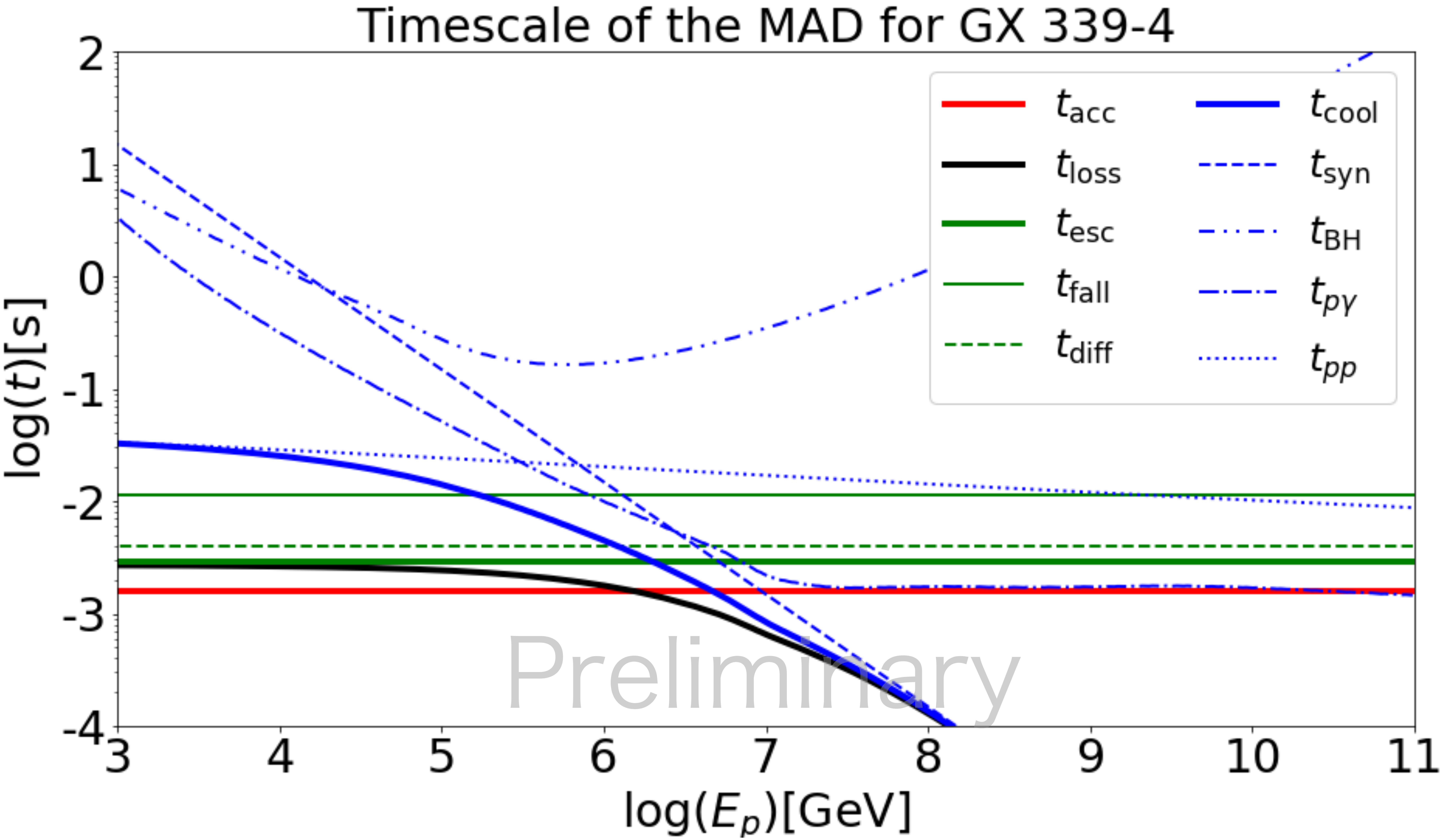
Stawarz&Petrosian 2008

$$\frac{\partial}{\partial \chi}\left(\chi^{q}\frac{\partial N}{\partial \chi}\right) - \frac{\partial}{\partial \chi}\left[\left(2\chi^{q-1} - \chi\vartheta_{\chi}\right)N\right] - \varepsilon\chi^{2-q}N = -Q.$$

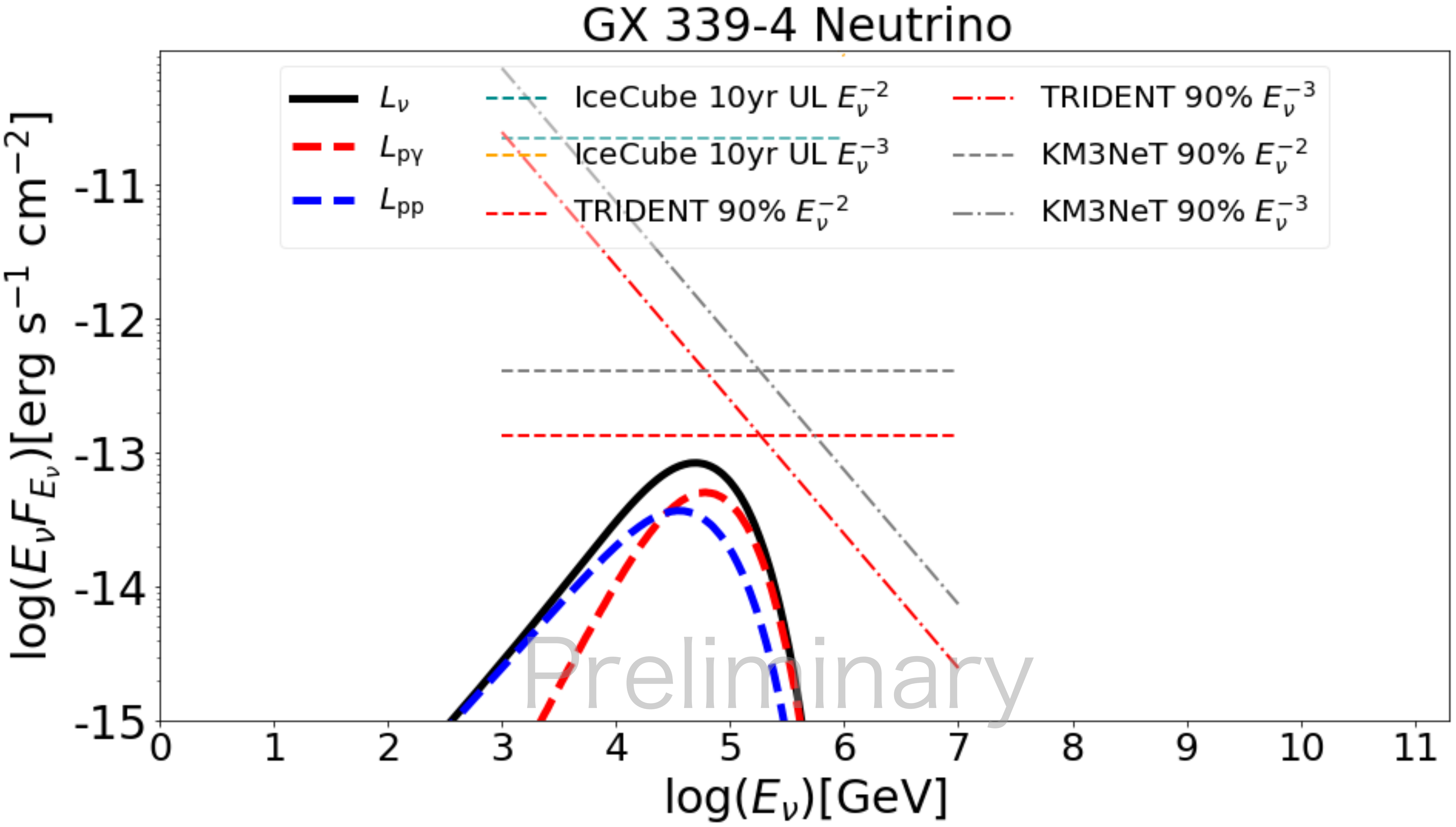
$$\chi = \frac{p}{p_0},\ \epsilon = \frac{t_{\rm acc}}{t_{\rm esc}},\ \vartheta = \frac{t_{\rm acc}}{t_{\rm cool}}$$

if $\vartheta \ll 1, N \propto \chi^{-s}$ -> $s = -\frac{1}{2} + \sqrt{\frac{9}{4} + \epsilon} \simeq 1.1$ for MAD model

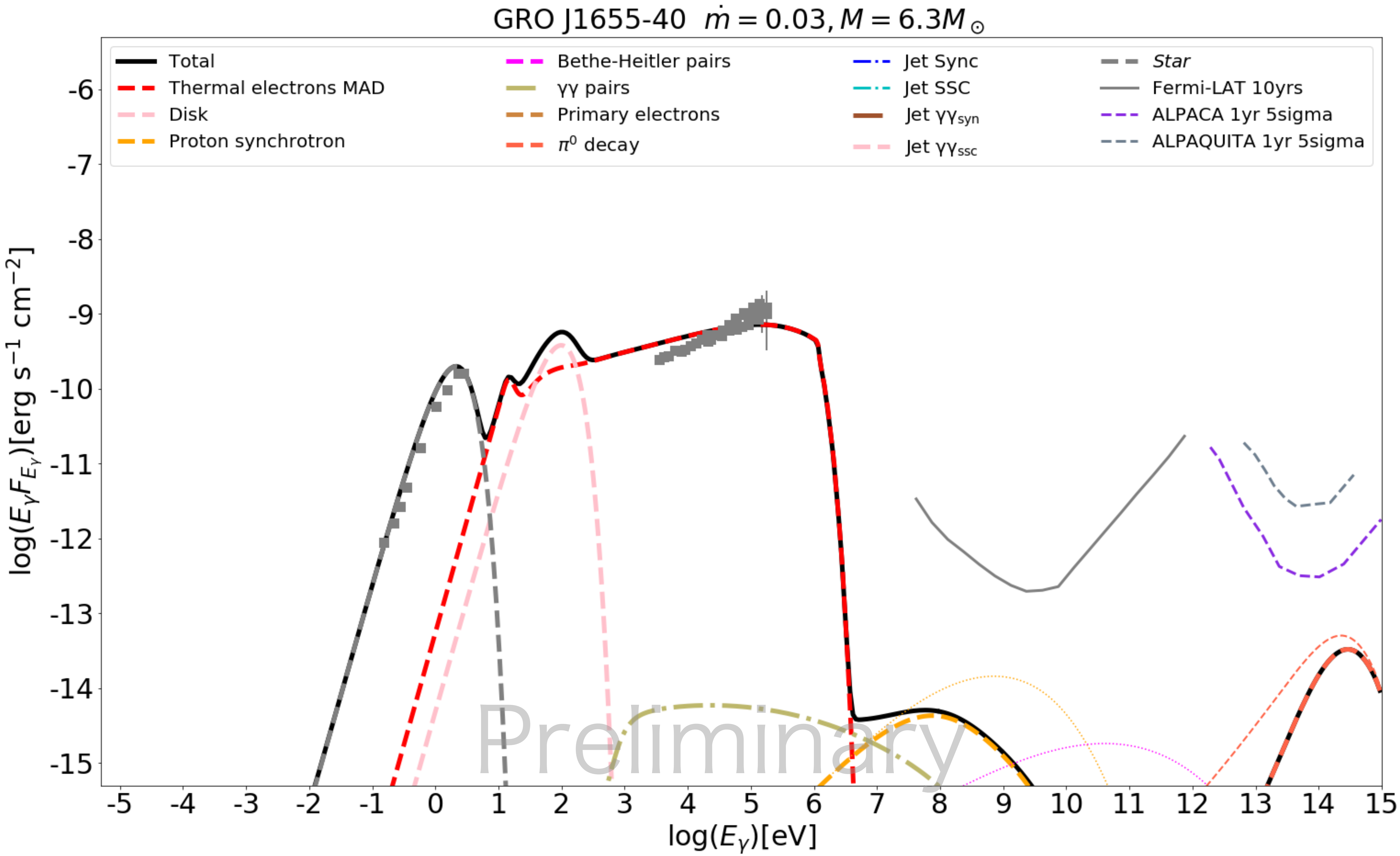
GX 339-4: Jet-MAD model



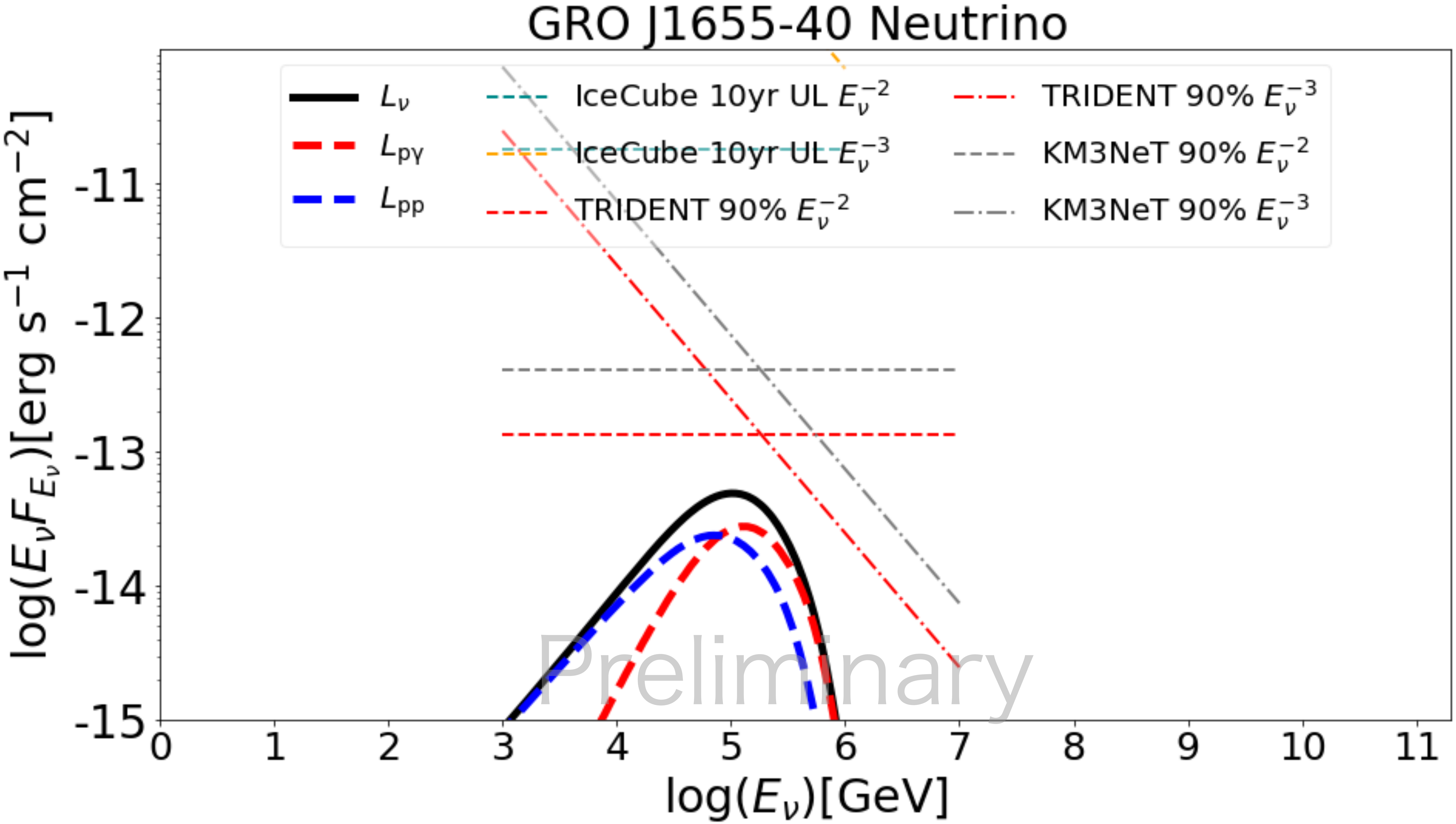
GX 339-4: Jet-MAD model



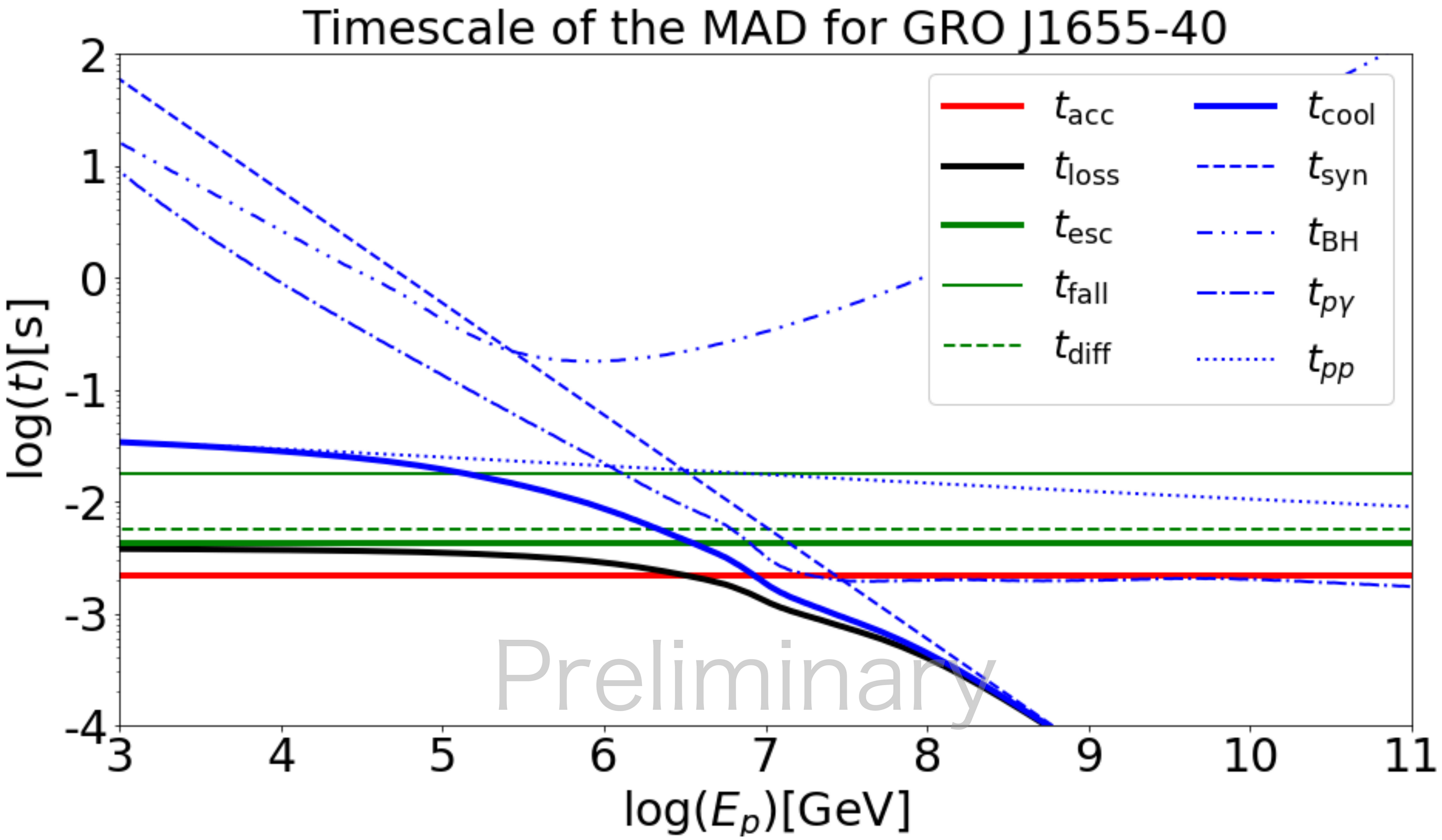
GRO J1655 with MAD parameters



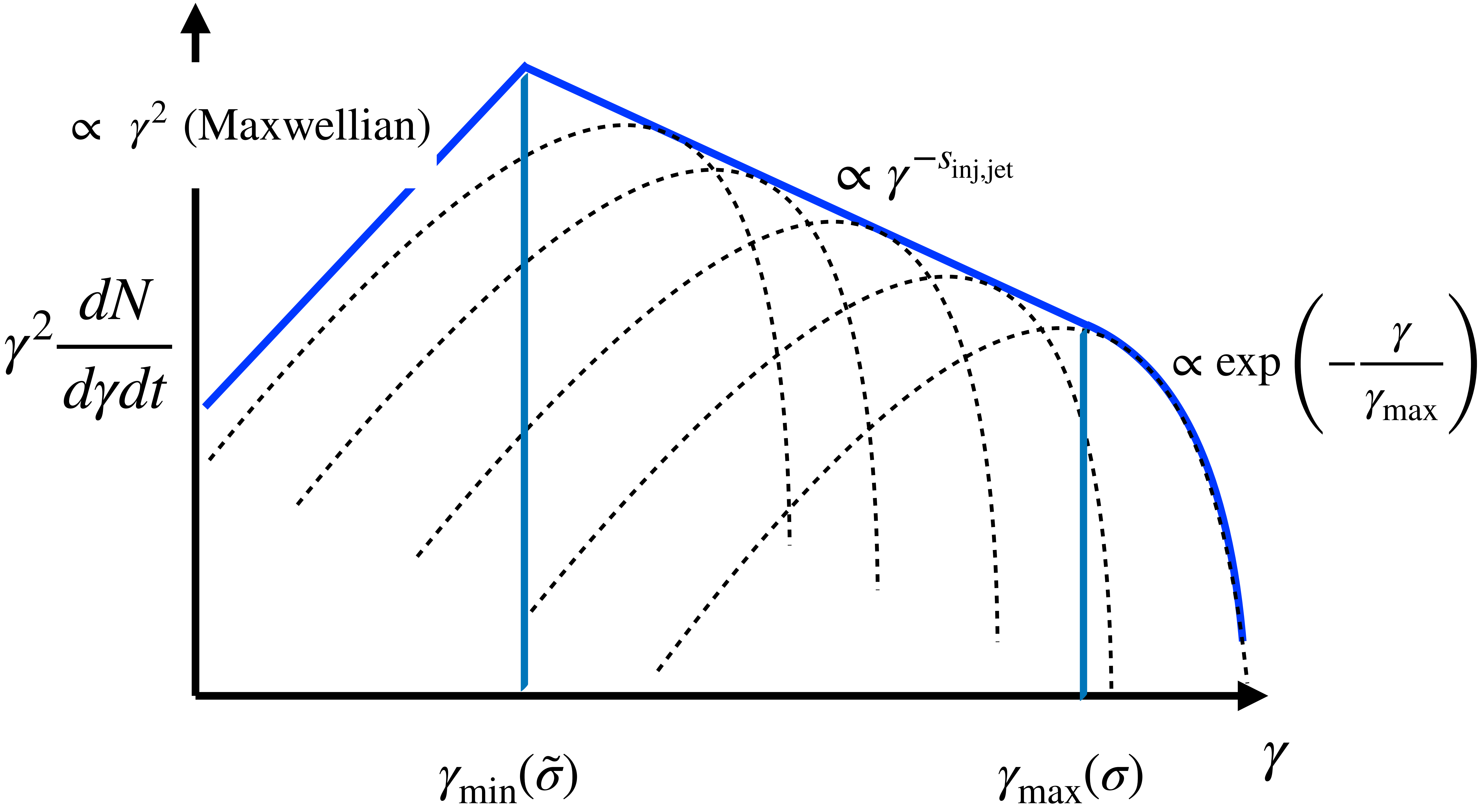
GRO J1655


GRO J1655-40 Neutrino
L_ν
$L_{p\gamma}$
L_{pp}
IceCube 10yr UL E_ν^{-2}
IceCube 10yr UL E_ν^{-3}
TRIDENT 90% E_ν^{-2}
TRIDENT 90% E_ν^{-3}
KM3NeT 90% E_ν^{-2}
KM3NeT 90% E_ν^{-3}
$\log(E_\nu F_{E_\nu})$[erg s^{-1} cm^{-2}]
-11
-12
-13
-14
-15
0
1
2
3
4
5
6
7
8
9
10
11
$\log(E_\nu)$[GeV]
Preliminary

GRO J1655


Timescale of the MAD for GRO J1655-40
$\log(t)[\mathrm{s}]$
2
1
0
-1
-2
-3
-4
3
4
5
6
7
8
9
10
11
$\log(E_p)[\mathrm{GeV}]$
$t_{\rm acc}$
$t_{\rm loss}$
$t_{\rm esc}$
$t_{\rm fall}$
$t_{\rm diff}$
$t_{\rm cool}$
$t_{\rm syn}$
$t_{\rm BH}$
$t_{p\gamma}$
t_{pp}
Preliminary

Particle distribution of Injection term



Maximum Energy

$$\gamma_{\max} = \xi \left(\frac{\delta B}{B}\right)^2 \sigma$$

δB: Amplitude of the perturbed magnetic field

ξ : The effect of the density reduction due to the expansion of the blob by the velocity dispersion of the blob

Minimum Energy

$$\gamma_{\min} = \left(\frac{\delta B}{B}\right)^2 \tilde{\sigma}$$