Accelerator Physics Basics

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for **US Muon Collider Accelerator School**
University of Chicago, August 3-6th 2025

Fermilab

Learn more at US Particle Accelerator School (USPAS)

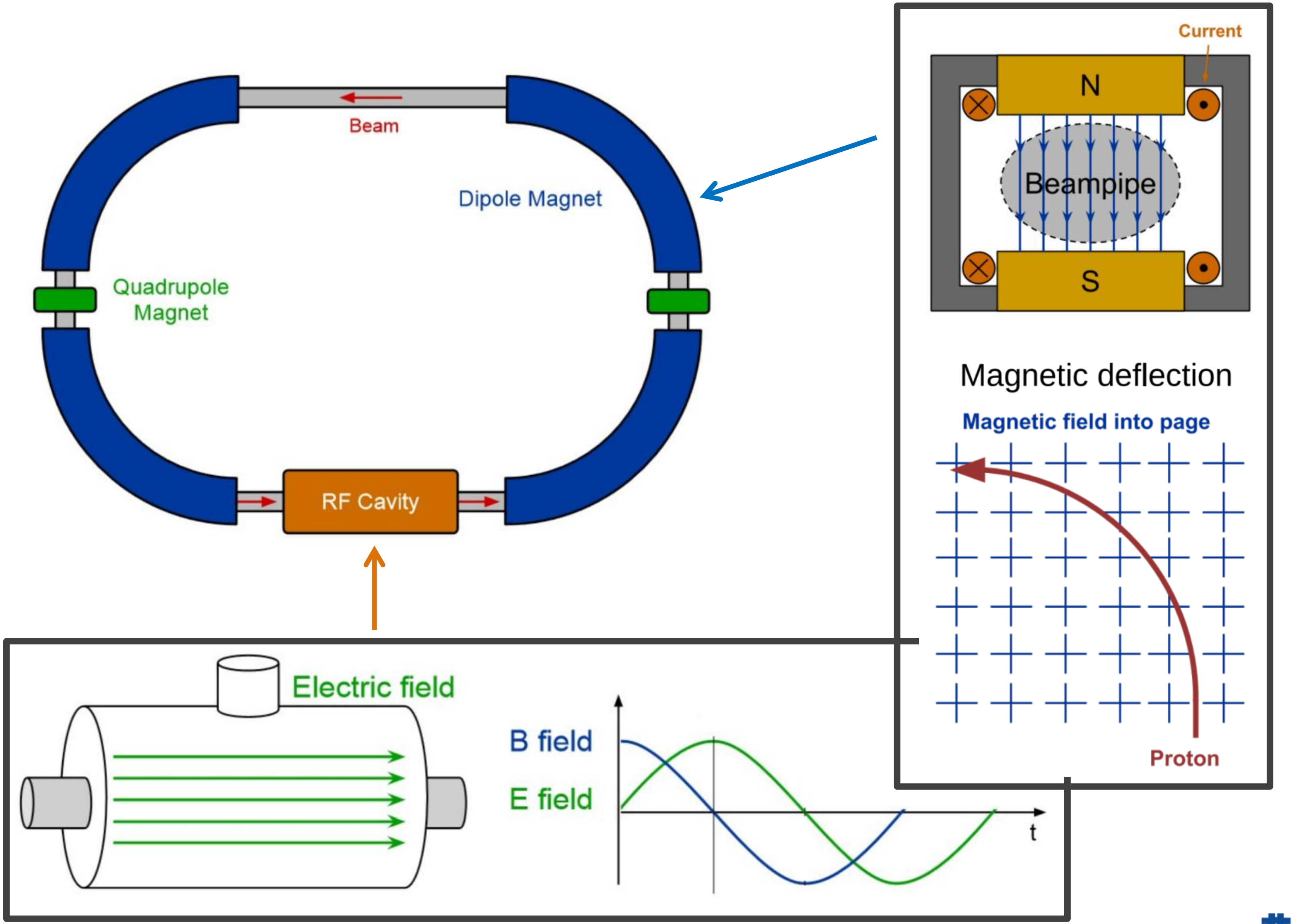
Free Recorded Classes:
- Eric Prebys' online course:
"Fundamentals of Accelerator Physics"
- Huang & I's online course:
"Mechanics & Electromagnetism for Accelerator Physics"

Textbook:
"An Introduction to the Physics of High Energy Accelerators"
Syphers and Edwards

Sign up for Live USPAS classes: website
- Two-week full-time sessions every June and January.
- Equivalent to graduate-level college-semester course!
- January 2023 session will be back to in-person (deadline Sept 15)

I took many USPAS classes as a graduate student, and now I regularly teach at USPAS.

Simplified Particle Accelerator



D. Barak, B. Harrison, A. Watts, Concepts Rookie Book, special thanks **A. Watts**

Dipole Magnets for Bending

Maximum Dipole Field -> Maximum Proton Energy

Lorentz Force:

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

Bending Radius in a constant dipole field:

$$B\rho = \frac{p}{Ze}$$

particle momentum

particle charge

Dipole field

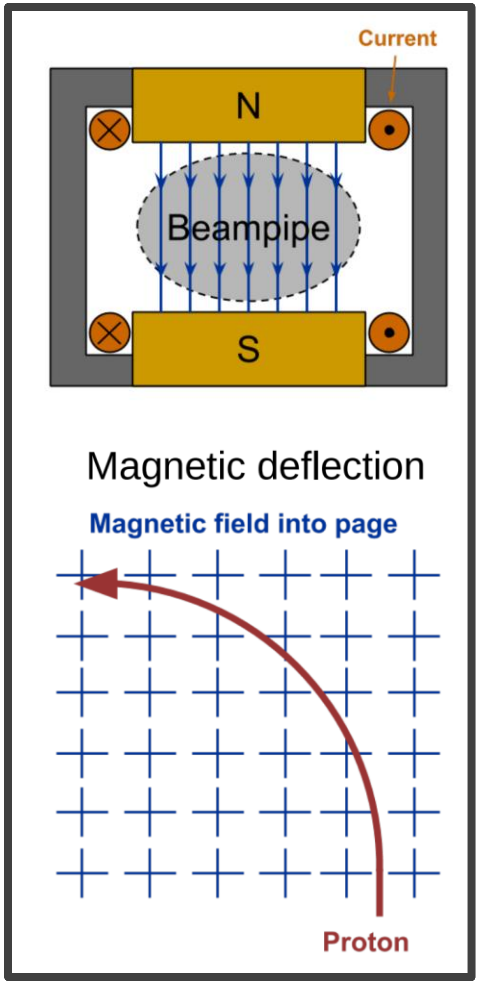
Bending radius

“Beam rigidity” Bρ in units of Tm

$$B\rho\ [\mathrm{Tm}] = 3.3357 p\ [\mathrm{GeV/c}]$$

Total bending for a ring is always 2π:

$$2\pi\rho = L_{dipole\ total}$$



Maximum Proton Energy – LHC Example

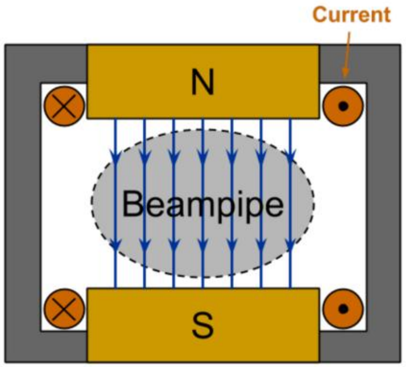
The LHC has **1232** dipoles, each **15m** long but effectively **14.3m** long.
- **18.5 km** of the **26.7 km** circumference is dipole magnet.
- We say “circumference” even though the LHC is a 1232-sided polygon.
- 1232*14.3/2π = **2800 m** magnetic bending radius ρ.

For a dipole field B of **7.7 T,** we can calculate the equivalent energy:
- 2800*7.7 / **3.3357** = 6,500 GeV/c proton
- **6.5 TeV** per beam
- **13 TeV** colliding energy

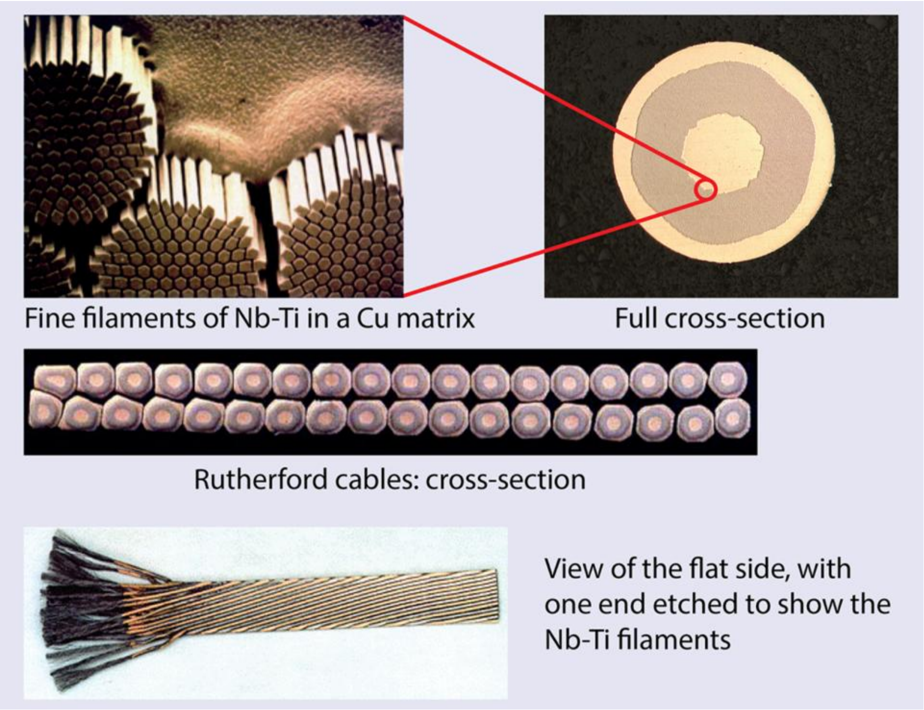
To go to higher energy:
- dig a longer tunnel
- and/or build a better dipole magnet:
  - superconductors have a critical temp, critical field, critical current.
  - manufacturing and reliability are part of dipole design as well.

LHC Superconducting Dipoles

Normal Conducting Dipole



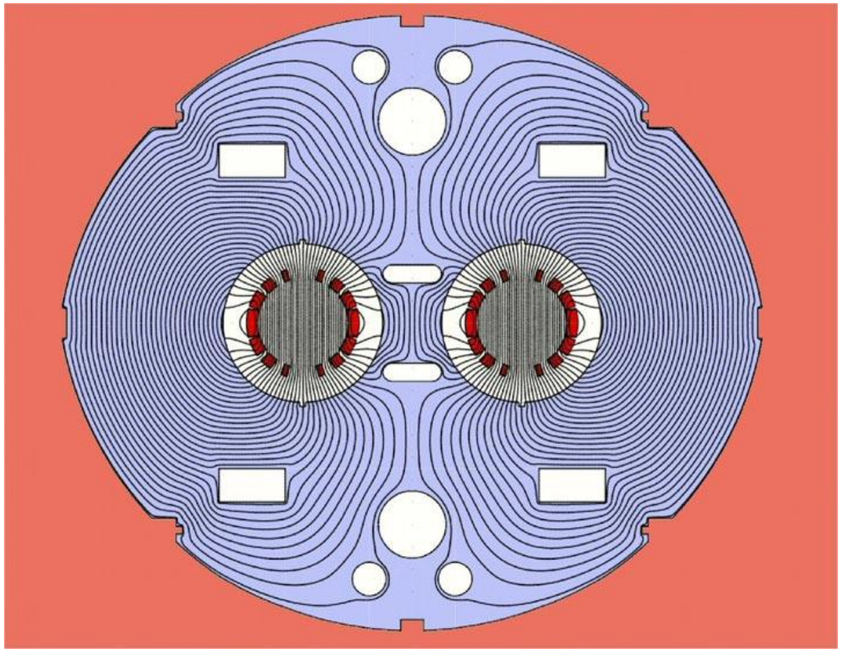
LHC Superconducting Dipole


Fine filaments of Nb-Ti in a Cu matrix

Full cross-section

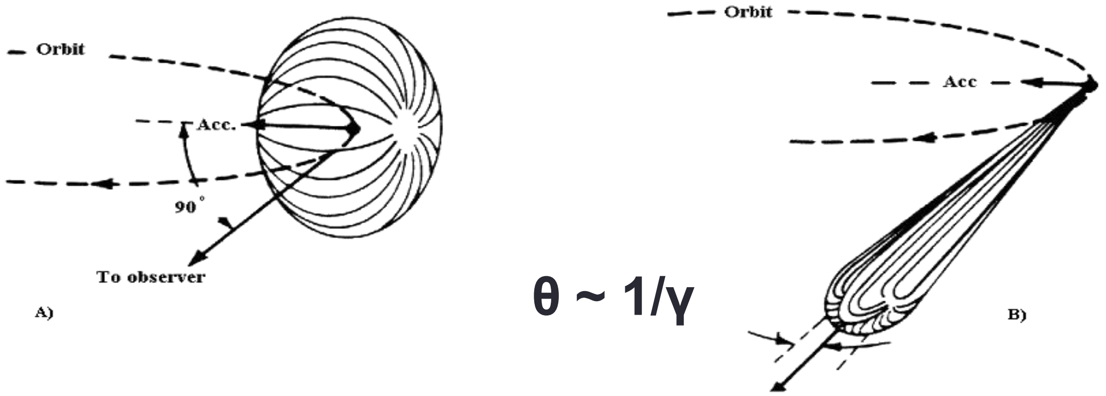
Rutherford cables: cross-section

View of the flat side, with one end etched to show the Nb-Ti filaments


CERN Courier

Radiation Loss -> Maximum Electron Energy

Circular electron colliders are limited instead by radiation.
Any relativistic charged particle will give off synchrotron radiation.



$\theta \sim 1/\gamma$

The magnitude of radiation is usually negligible in hadron machines, but is a dominant feature of electron machines.

$$P = \frac{q^2 c}{6\pi\epsilon_0} \frac{\beta^4\gamma^4}{\rho^2}$$

Relativistic βγ factors

Bending radius

Power radiated

Physical constants

The maximum acceleration rate sets the maximum power loss.
For x2 the energy, the same power loss occurs at x4 the bending radius.

Power Radiated from Accelerating Charge

Power Radiated (per particle):

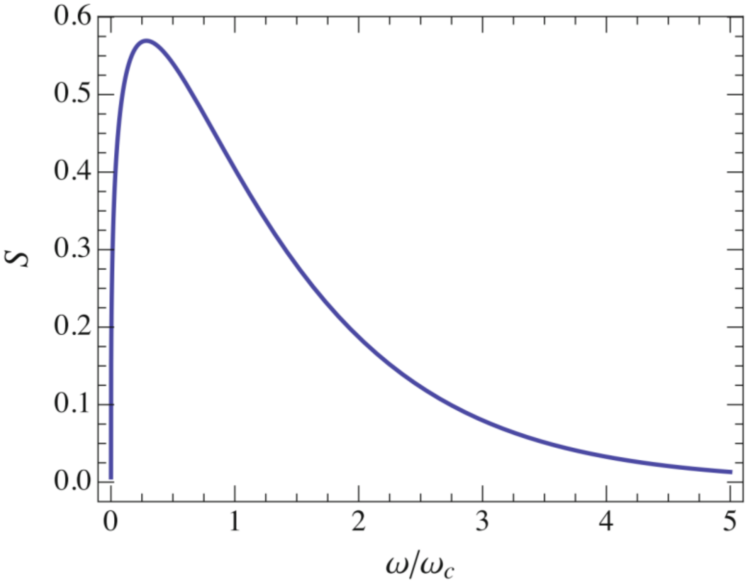
$$P = \frac{q^2 c}{6\pi\epsilon_0}\frac{\beta^4\gamma^4}{\rho^2} = [4.6\text{e-}20\ \mathrm{J\ m^2/s}]\frac{\gamma^4}{\rho} = [0.29\ \mathrm{eV\ m^2/s}]]\frac{\gamma^4}{\rho}$$

Energy Loss per revolution (per particle):

$$dE = \frac{q^2}{3\epsilon_0}\frac{\beta^4\gamma^4}{\rho} = [9.7\text{e-}28\ \mathrm{J\ m}]\frac{\gamma^4}{\rho} = [6.0\text{e-}9\ \mathrm{eV\ m}]]\frac{\gamma^4}{\rho}$$

Critical Frequency:

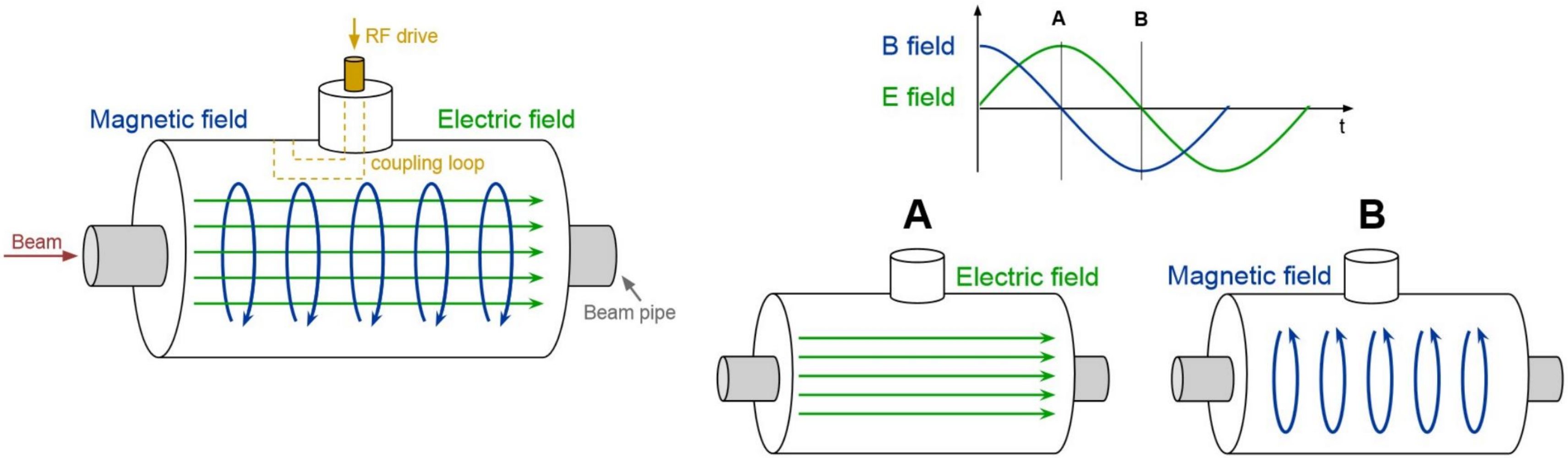
$$\omega_c = \frac{3c\gamma^3}{2\rho}$$



RF Cavities
for Acceleration
(and Synchronization)

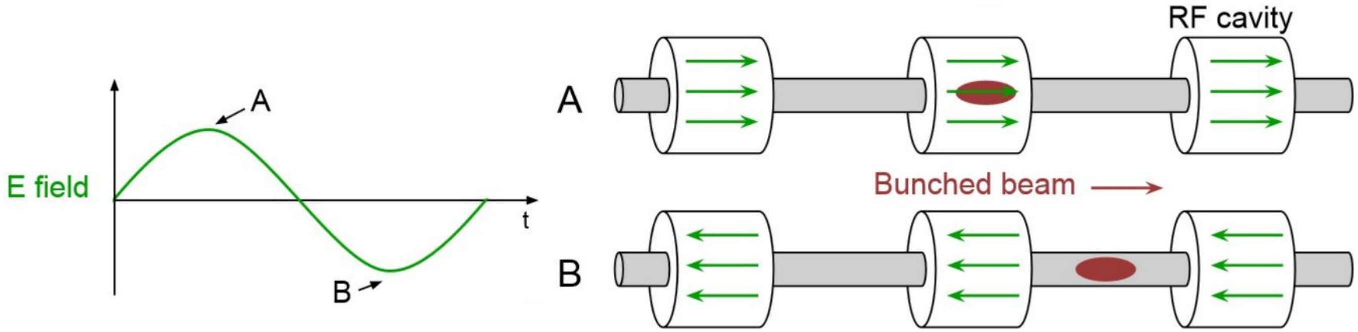
RF Accelerating Cavity

We use resonating radiofrequency (RF) cavities to efficiently trap an electromagnetic wave which accelerates the beam.



A. Watts, Concepts Rookie Book

The beam must arrive in synchronized bunches to be accelerated.



Change in Momentum

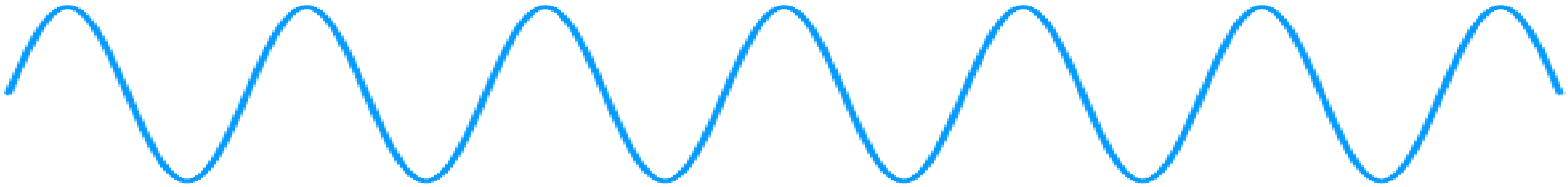
Fractional Momentum: $\delta \equiv \frac{p - p_0}{p_0}$

RF Acc. Per Pass: $\Delta E = qV \sin(\phi)$

Change Momentum per unit time:

$$\dot{\delta} = \frac{\dot{p}}{p_0} = \frac{\dot{E}}{\beta^2 E_0} = f_{rev} \frac{\Delta E}{\beta^2 E_0} = f_{rev} \frac{qV}{\beta^2 E_0} \sin(\phi)$$

Sinesoidal potential:



Phase-Slip Factor η

The arrival time of the particle depends on the momentum:

$$\delta \equiv \frac{p - p_0}{p_0} \qquad\qquad \frac{T - T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} \delta = \eta \delta$$

Higher momentum particles may arrive earlier or later than lower momentum particles:

$$\eta = \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} = \frac{1}{C} \frac{\partial C}{\partial \delta} - \frac{1}{\beta} \frac{\partial \beta}{\partial \delta} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}$$

Momentum compaction factor:

$$\text{where, } \alpha_c = \frac{1}{C} \frac{\partial C}{\partial \delta} = \frac{1}{C} \int_0^C \frac{D(s)}{\rho} ds$$

We can write the change in phase per unit time using the phase-slip factor:

$$\dot{\phi} = f_{rev} \Delta \phi = 2\pi f_{rev} \frac{\Delta T}{T_{rf}} = 2\pi f_{rev} h \frac{\Delta T}{T_{rev}} = 2\pi f_{rev} h \eta \delta$$

$$f_{rf} = h f_{rev}$$

Longitudinal Focusing

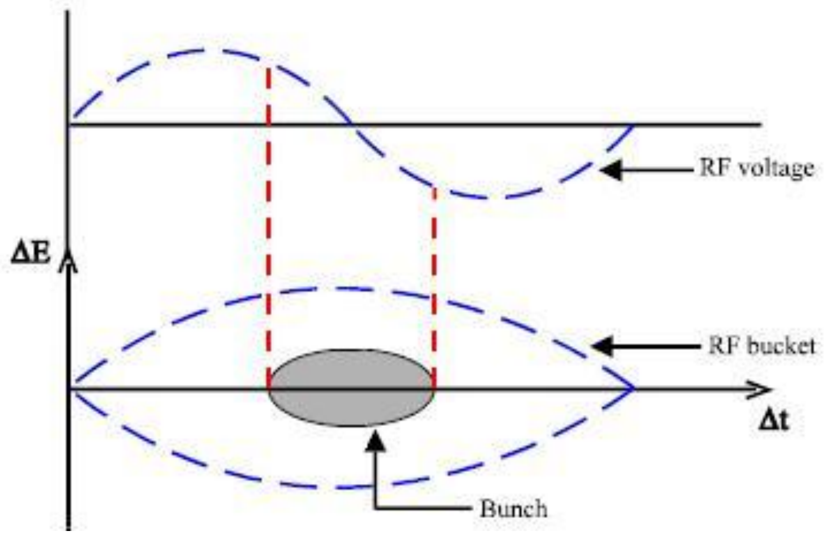
$$\dot{\phi} = 2\pi f_{rev} h \eta \delta, \;\; \dot{\delta} = f_{rev} \frac{qV}{\beta^2 E_0} \sin(\phi)$$

$$\ddot{\phi} = 2\pi f_{rev}^2 \frac{qV}{\beta^2 E_0} h\eta \sin(\phi)$$

$$\eta < 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi)$$
$$\eta > 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi + \pi)$$

Synchrotron Tune

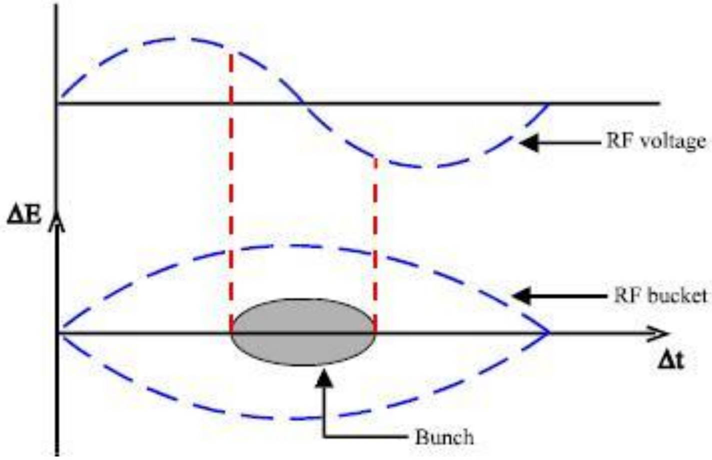
$$\omega_s = 2\pi f_{rev} \sqrt{\frac{qVh|\eta|}{2\pi\beta^2 E_0}}$$

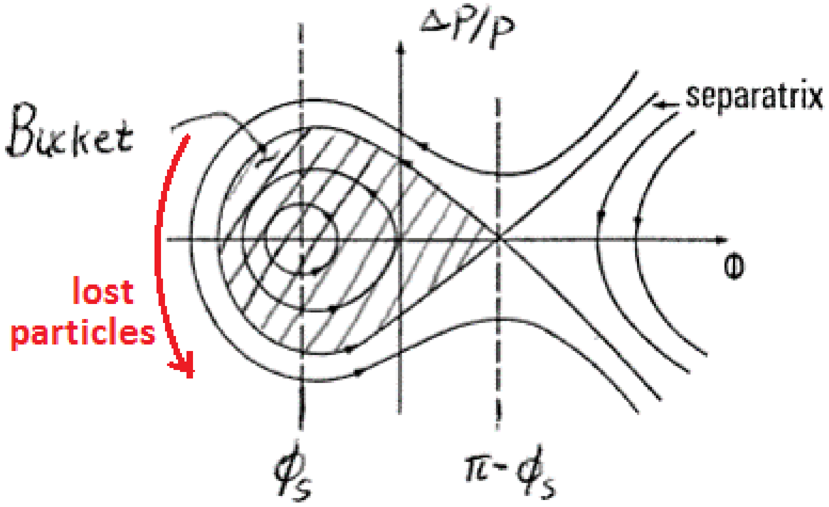


RF Acceleration

A fixed frequency beam longitudinally focuses the beam into a several beam “bunches” in individual RF “buckets”.

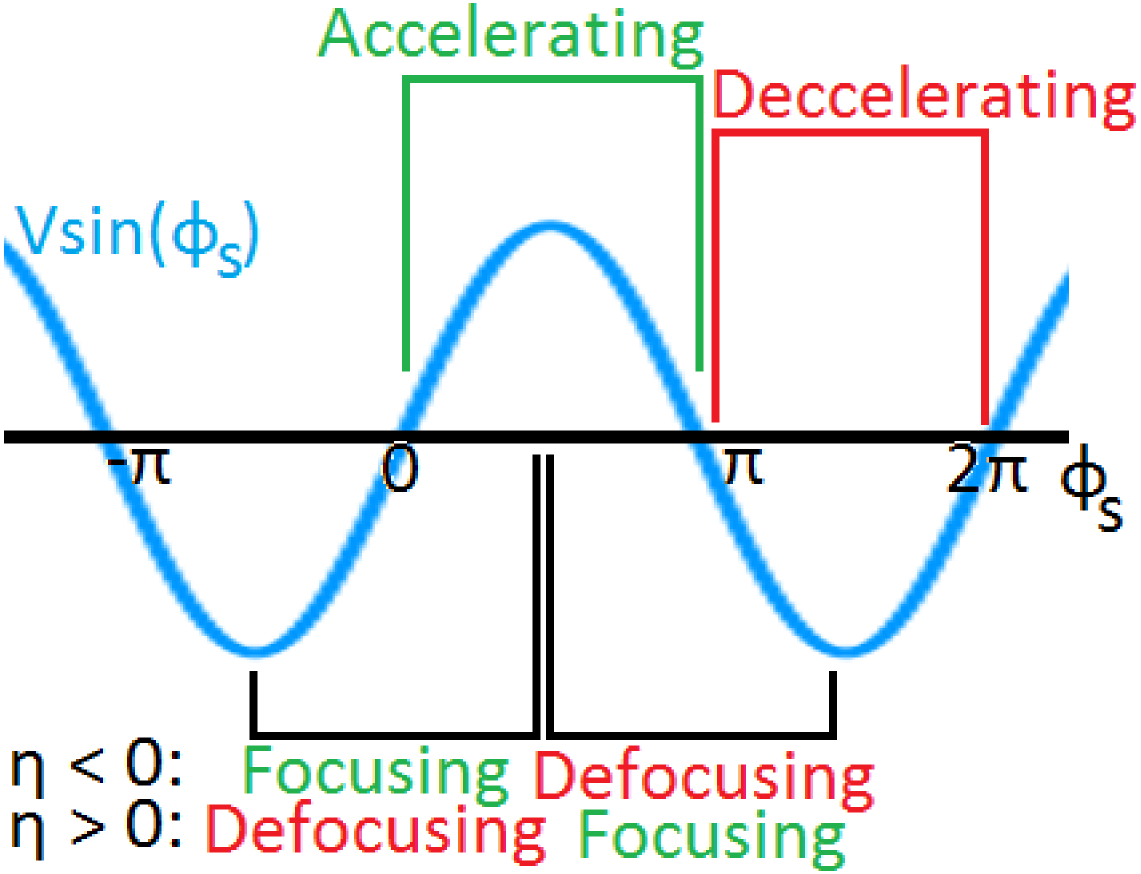
Particles in the bucket can be accelerated by adiabatically changing the RF frequency, the other particles are lost.





$$\dot{\delta} = f_{rev} V_\delta [\sin(\phi) - \sin(\phi_s)], \;\; \dot{\phi} = 2\pi f_{rev} h \eta \delta$$

Phase-Focusing & Acceleration

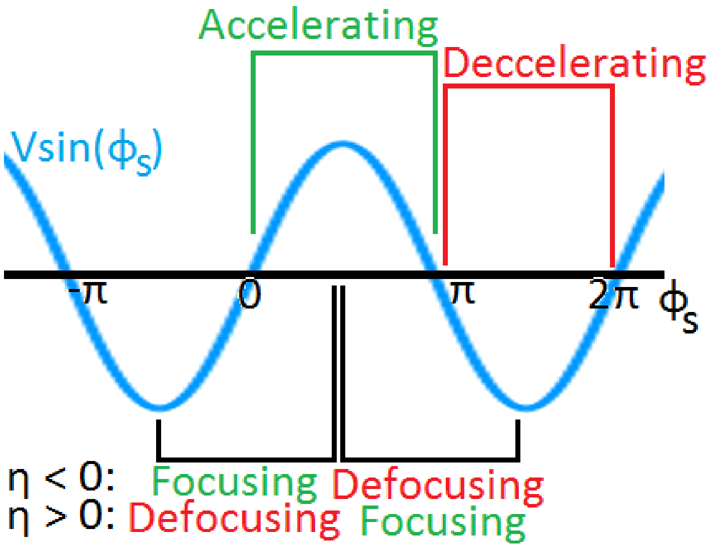


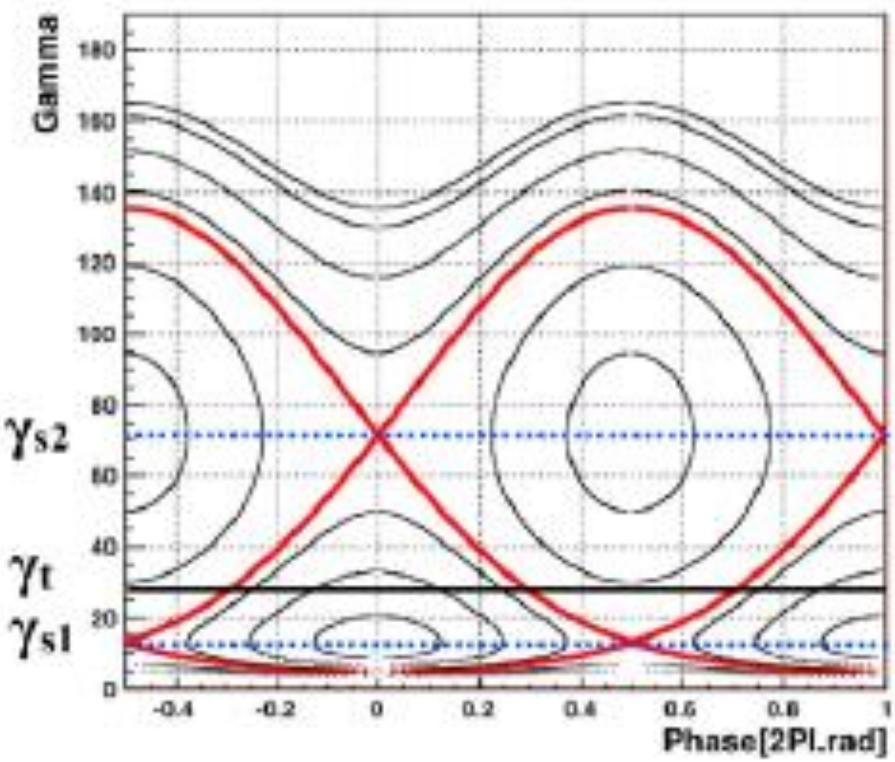
$$\dot{\delta} = f_{rev} V_{\delta}[\sin(\phi) - \sin(\phi_s)], \;\; \dot{\phi} = 2\pi f_{rev} h \eta \delta$$

$$\eta < 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi)$$

$$\eta > 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi + \pi)$$

Phase-space at Transition

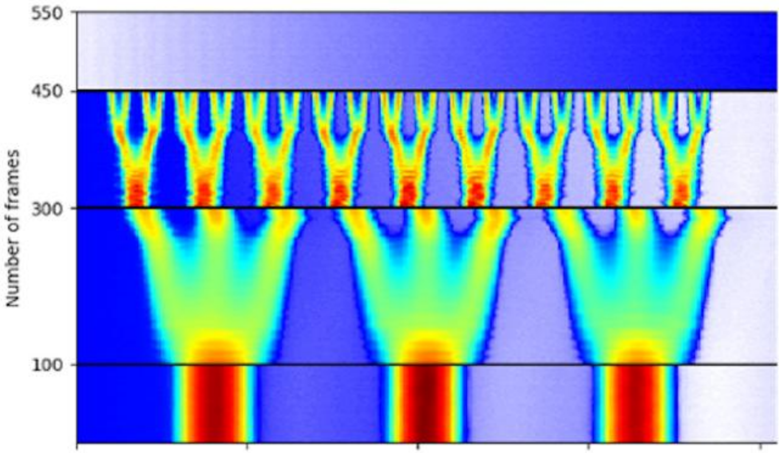




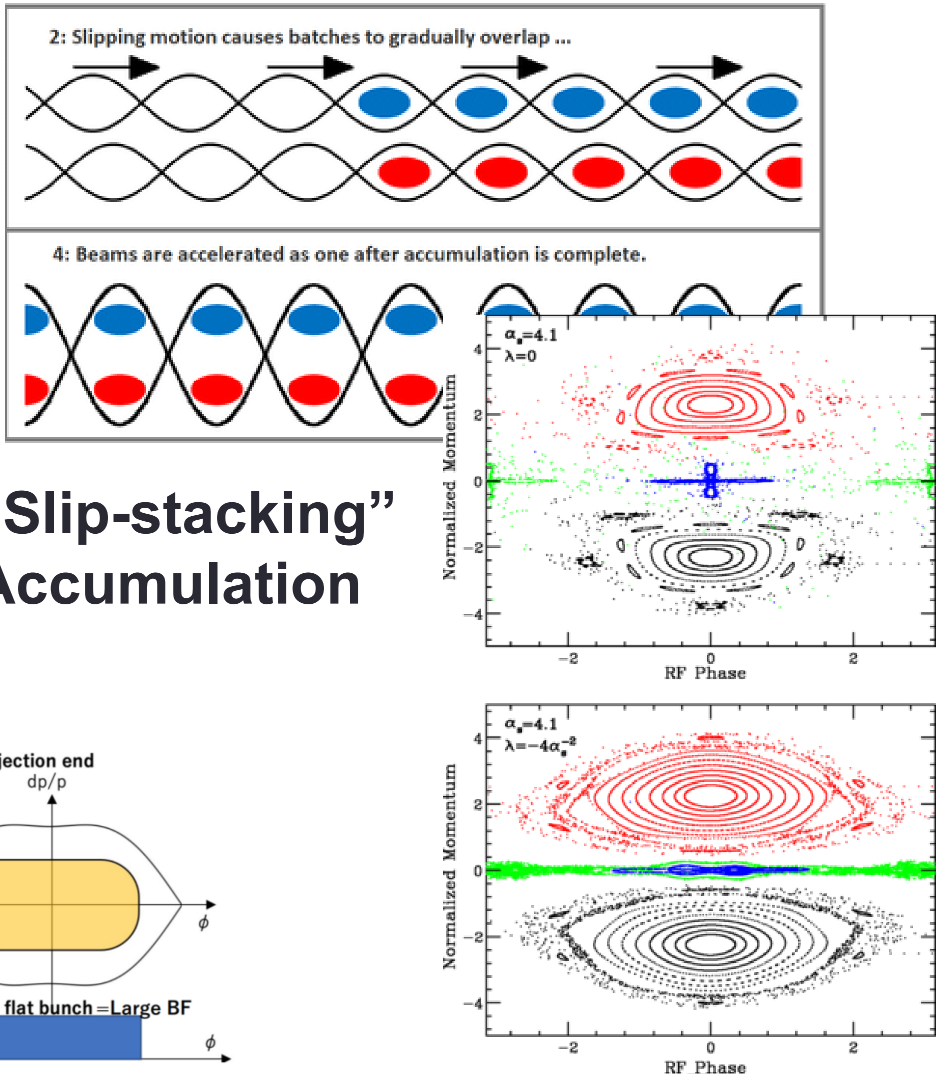
$$\frac{T-T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}}\frac{\partial T}{\partial \delta}\delta + \frac{1}{T_{rev}}\frac{\partial^2 T}{\partial \delta^2}\frac{\delta^2}{2} = \eta_0\delta + \eta_1\delta^2$$

Longitudinal Dynamics is itself a rich topic…

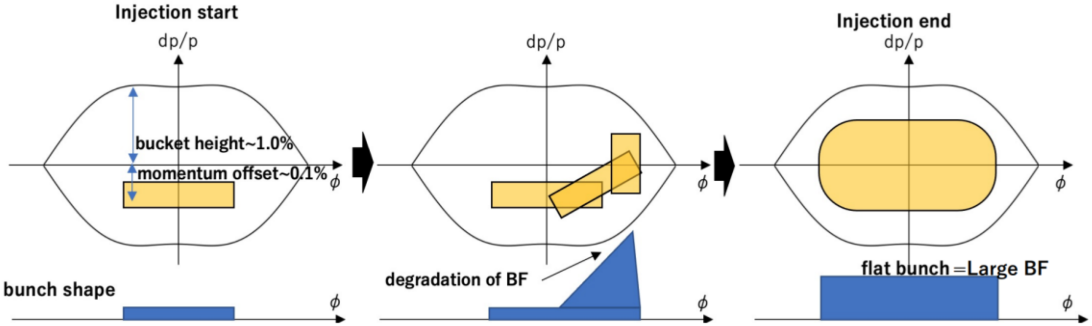
Bunch Splitting:



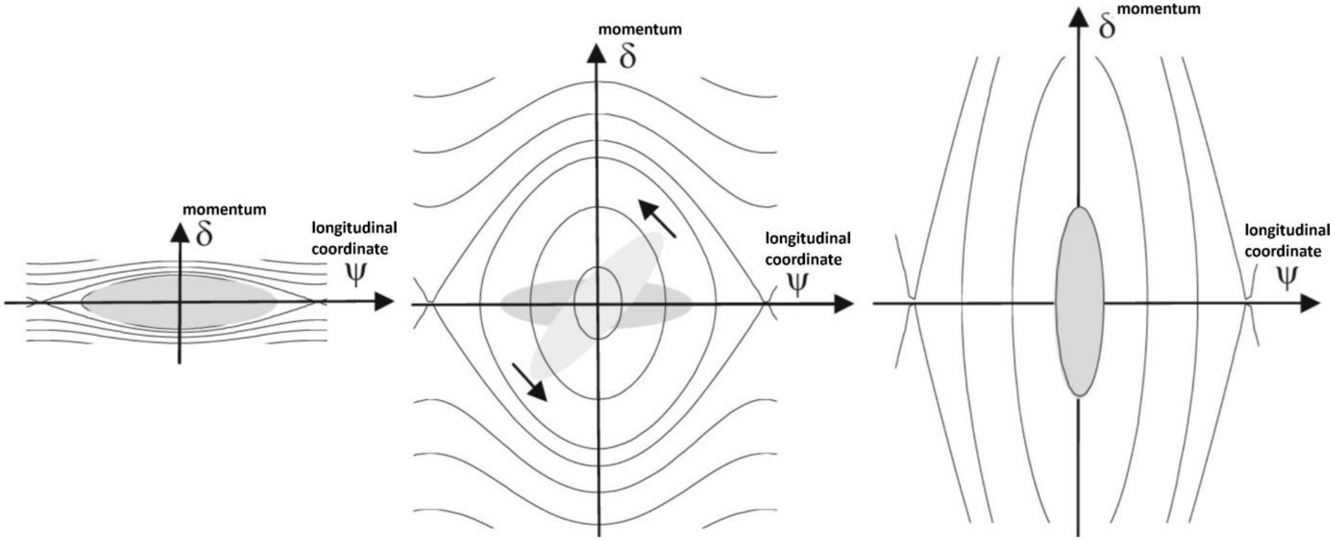
"Slip-stacking" Accumulation



Bunch Flattening & Longitudinal Painting:

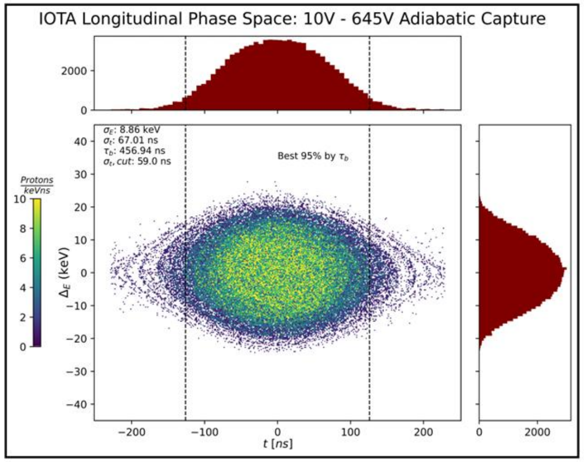


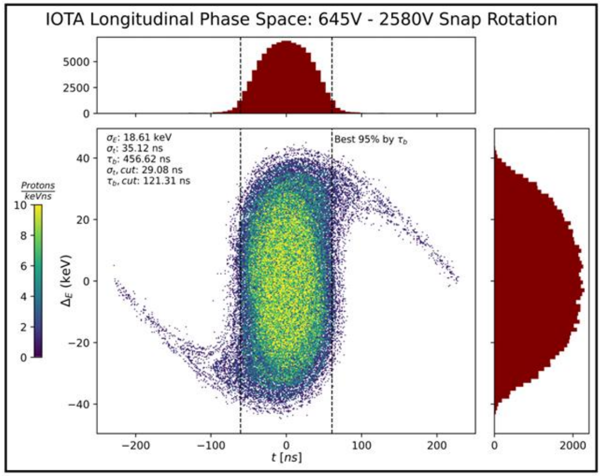
Bunch Rotation is Critical for Muon Collider



Low momentum spread exchanged for low temportal spread!

Wiedemann Textbook

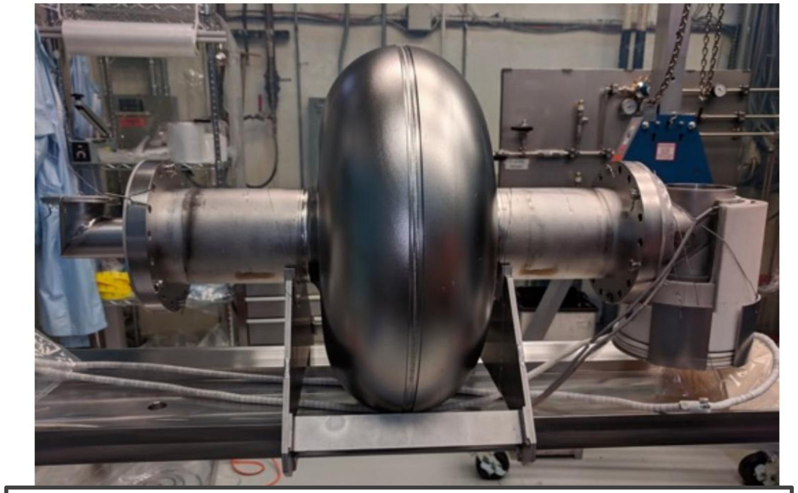


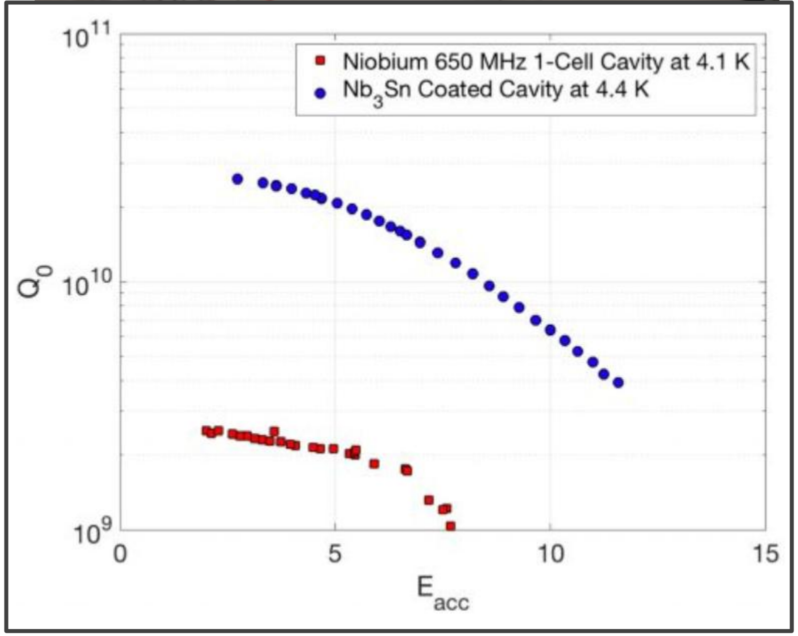


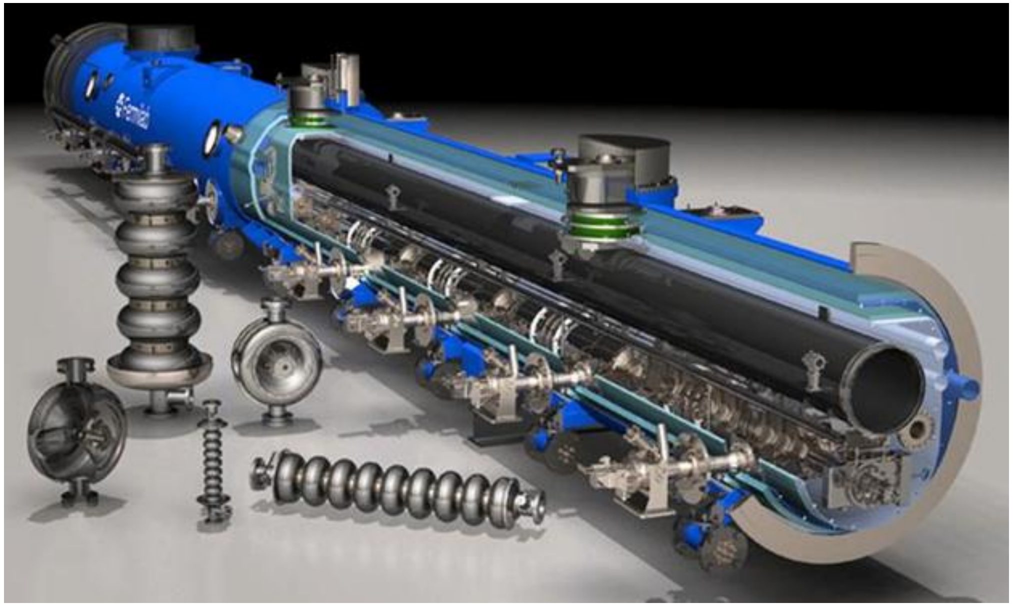
Ben Simons Simulation

Superconducting RF R&D

S. Posen, 2018







Pictures from FNAL PIP-II SRF.

Optimize Q-factor and Eacc (MV/m)

$$Q = \omega_0 \frac{U}{P_{\text{loss}}}$$

Stored energy

RF Power loss

Quadrupole Magets for Transverse Focusing

Luminosity

Luminosity is proportional to the number of particle interactions in colliding beams, which (to lowest order) is given by:

Particles per bunch

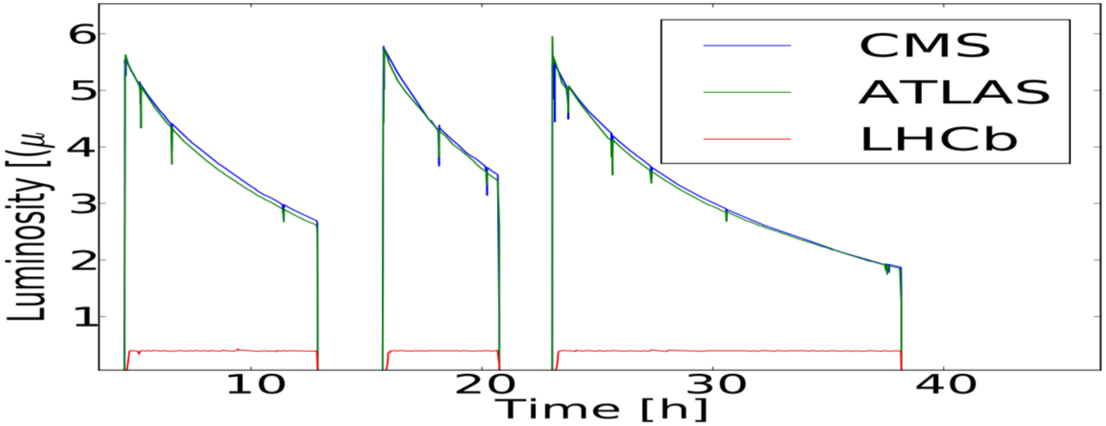
Luminosity → $$\mathcal{L} \approx \frac{N_1 N_2 f n_b}{4\pi\sigma_x\sigma_y}$$

Rate of bunch crossing

Horz, Vert beam sizes

Luminosity benefits from achieving the highest possible particle density in the beams, transversely and longitudinally.

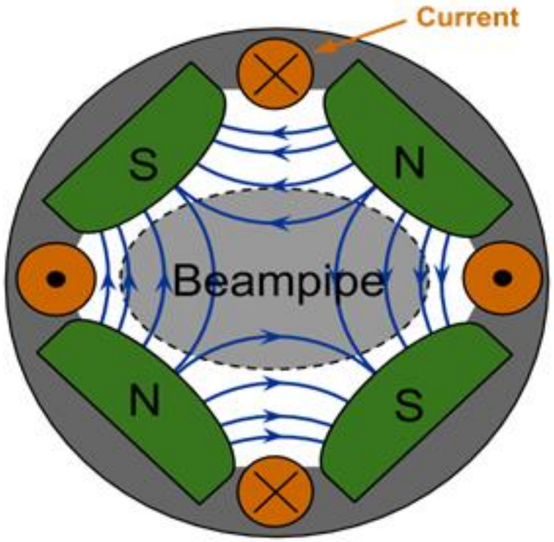
Luminosity typically degrades over time.



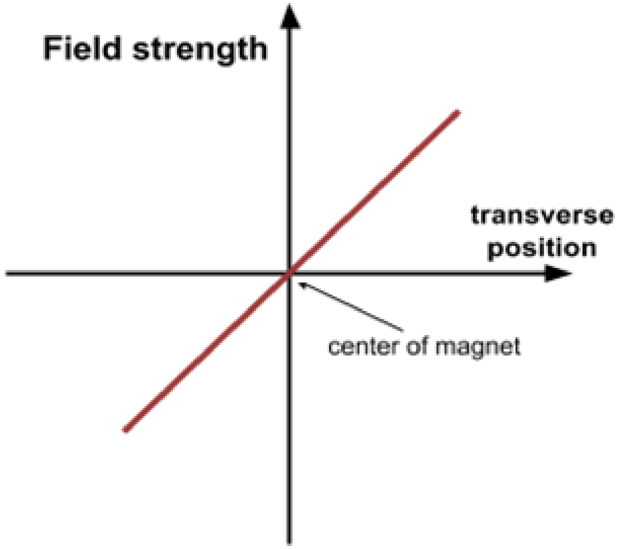
X. Buffat

Quadrupole Magnets for Transverse Focusing

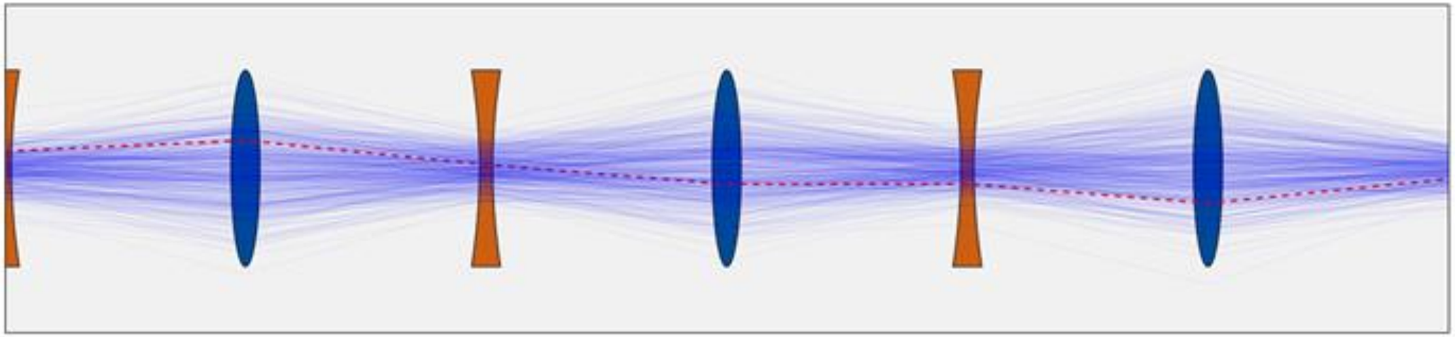
Quadrupole Magnet:



Linear Restoring Force:



Alternating Focusing Magnets:



D. Barak, B. Harrison, A. Watts, Concepts Rookie Book, special thanks A. Watts

Transverse “Betatron” Motion

Harmonic Oscillator

Hamiltonian: $H = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2$

Equations of motion:

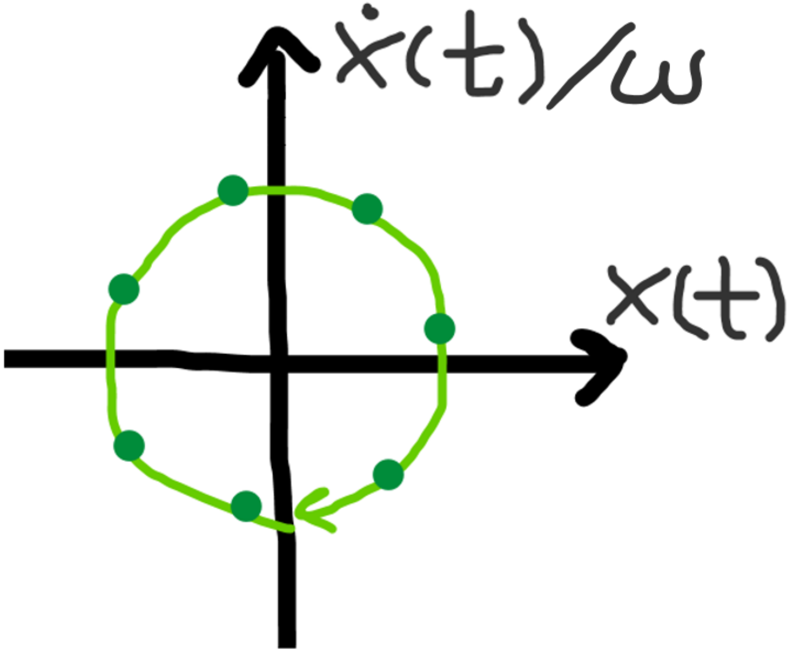
$$m\ddot{x} = -kx$$

$$\ddot{x} = -\omega^2 x$$

$$x(t) = A\cos(\omega t + \phi)$$

$$\dot{x}(t) = -\omega A \sin(\omega t + \phi)$$

Phase-space diagram:



Harmonic Oscillator

Hamiltonian: $H = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2$

Equations of motion, with action:

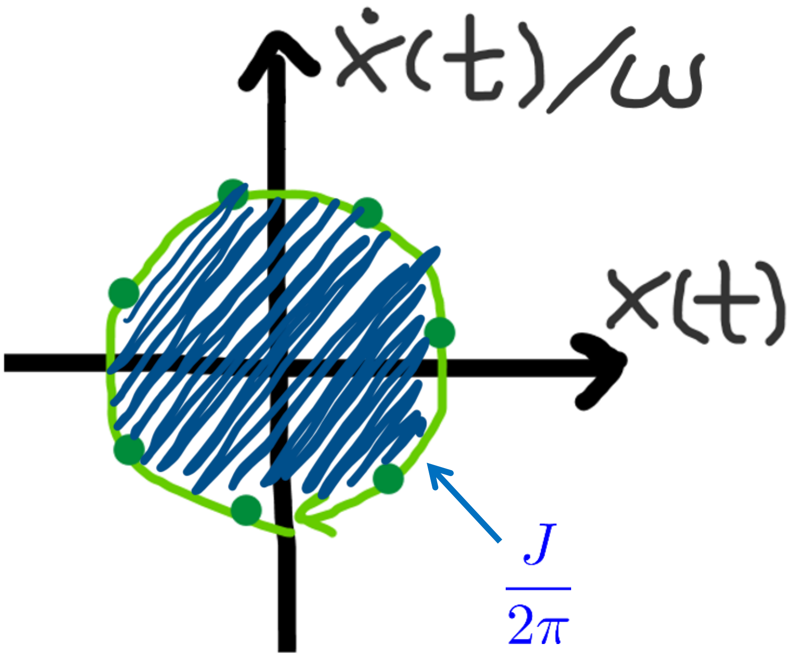
$$\ddot{x} = -\omega^2 x$$

$$x(t) = \sqrt{2J}\cos(\omega t + \phi)$$

$$\dot{x}(t) = -\sqrt{2J}\omega\sin(\omega t + \phi$$

$$x^2 + (\dot{x}/\omega)^2 = 2J$$

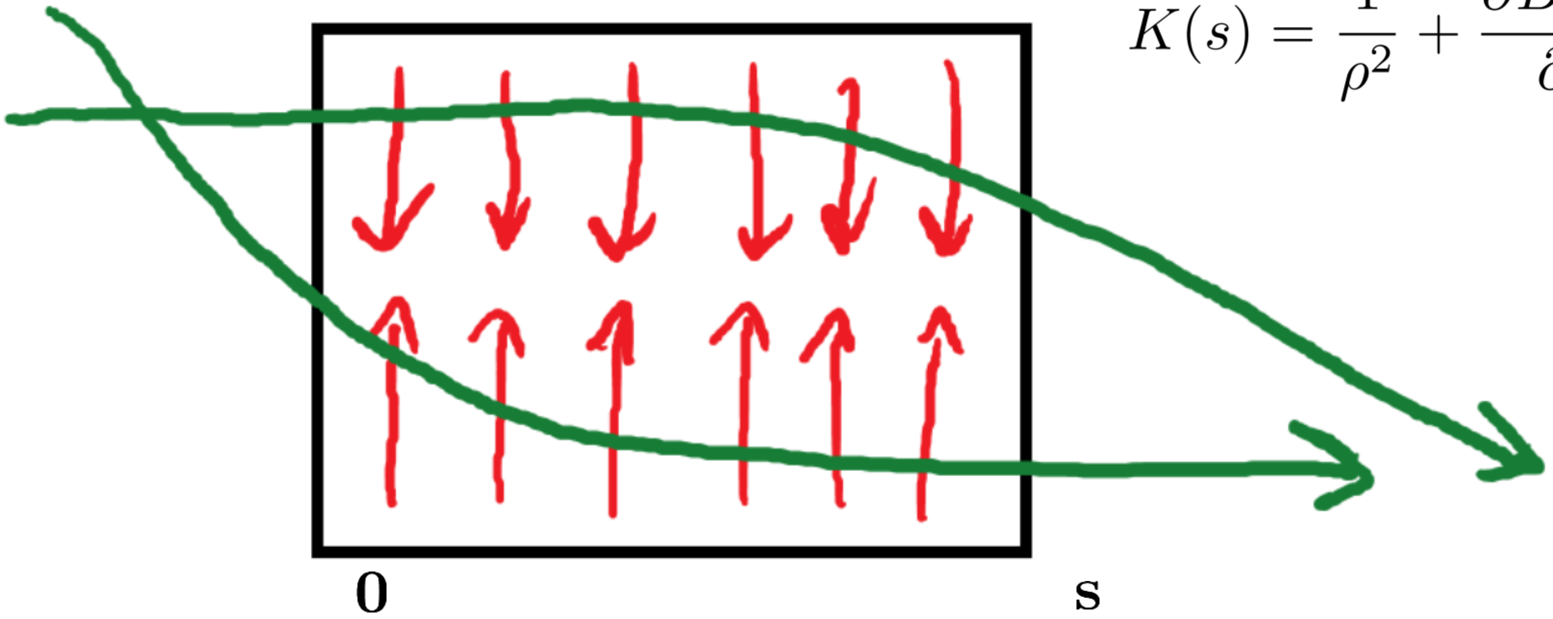
Phase-space diagram:



Linear Focusing

We can solve the linear Hill's equation: $x'' + K(s)x = 0$

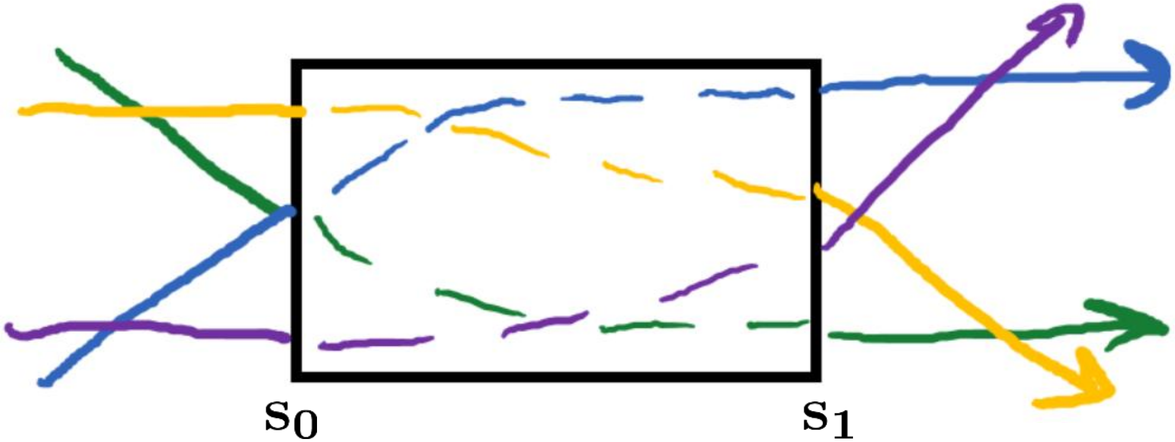
$$K(s) = \frac{1}{\rho^2} + \frac{\partial B_z(s)}{\partial x}\frac{1}{B\rho}$$



$$x(s) = \begin{cases} x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}}\sinh(\sqrt{|K|}s), & K < 0 \\ x_0 + x_0's, & K = 0 \\ x_0 \cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}}\sin(\sqrt{K}s), & K > 0 \end{cases}$$

Transfer Matrices

$$x(s) = \begin{cases} x_0 \cosh(\sqrt{|K|}s) + \frac{x'_0}{\sqrt{|K|}} \sinh(\sqrt{|K|}s), & K < 0 \\ x_0 + x'_0 s, & K = 0 \\ x_0 \cos(\sqrt{K}s) + \frac{x'_0}{\sqrt{K}} \sin(\sqrt{K}s), & K > 0 \end{cases}$$

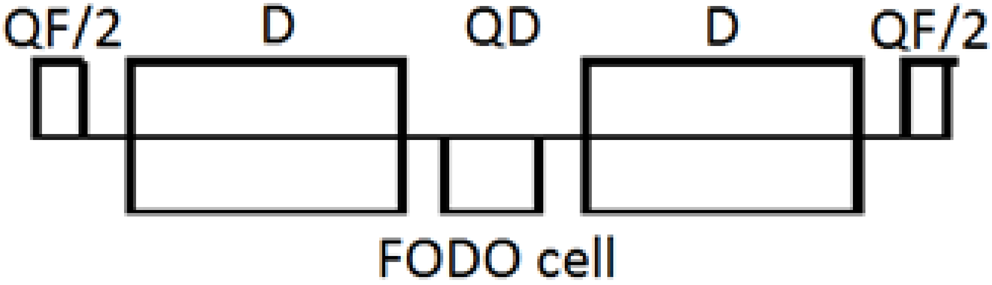


The final position and slope is a linear combination of the initial position and initial slope. We can use matrices:

$$x' = \frac{p_x}{p_0}$$

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

Example: FODO Cell



FODO cell

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$

The transfer matrix for a sequence of elements can be obtained by multiplying the matrices for the components.

This is good for tracking particles, but how can we make sense of what is happening to the beam as a whole?

Horizontal and Vertical Motion

Horizontally focusing quadrupoles are vertically defocusing.
Horizontally defocusing quadrupoes are vertically focusing.

$$\text{if } M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \text{ then } M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

In general the combined effect can be written as a 4x4 matrix:

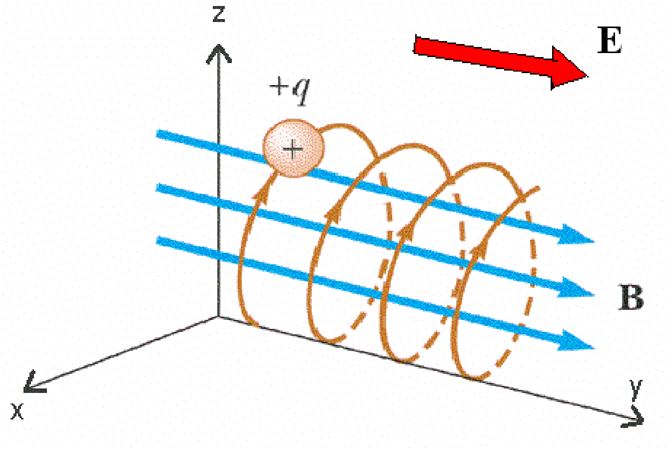
$$\left(\begin{array}{c} x_1 \\ x_1' \\ \hline y_1 \\ y_1' \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 1 \end{array}\right) \left(\begin{array}{c} x_0 \\ x_0' \\ \hline y_0 \\ y_0' \end{array}\right)$$

When there are skew quads (quads rotated 45 degrees) or magnets with longitudinal magnetic fields (like solenoids), Those off-diagonal blocks might be nonzero.

Solenoid Field

In the body of a solenoid we have: $k = \frac{B_{||}}{B\rho}$

$$\begin{pmatrix} x_1 \\ x_1' \\ \hline y_1 \\ y_1' \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & \frac{\sin(kL)}{k} & 0 & \frac{1-\cos(kL)}{k} \\ 0 & \cos(kL) & 0 & \sin(kL) \\ \hline 0 & -\frac{1-\cos(kL)}{k} & 1 & \frac{\sin(kL)}{k} \\ 0 & \text{-}\sin(kL) & 0 & \cos(kL) \end{array}\right) \begin{pmatrix} x_0 \\ x_0' \\ \hline y_0 \\ y_0' \end{pmatrix}$$



Recall…

$$B\rho = \frac{p}{Ze}$$

Larmor radius

$$\rho = \frac{B\rho}{B_{||}} x' = 3.3357 \frac{p_\perp \text{ [GeV]}}{B_{||} \text{ [GeV]}}$$

Solving Hill’s Equation (via Floquet method)

Hill’s Equation: $x'' + K(s)x = 0$

If we write: $x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)]$

$$x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\Big[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\Big]$$

This is a solution if we also require that:

$$\alpha_x(s) = -\frac{1}{2}\beta_x'(s)$$

$$\phi_x(s) = \int_0^s \frac{1}{\beta_x(z)}dz$$

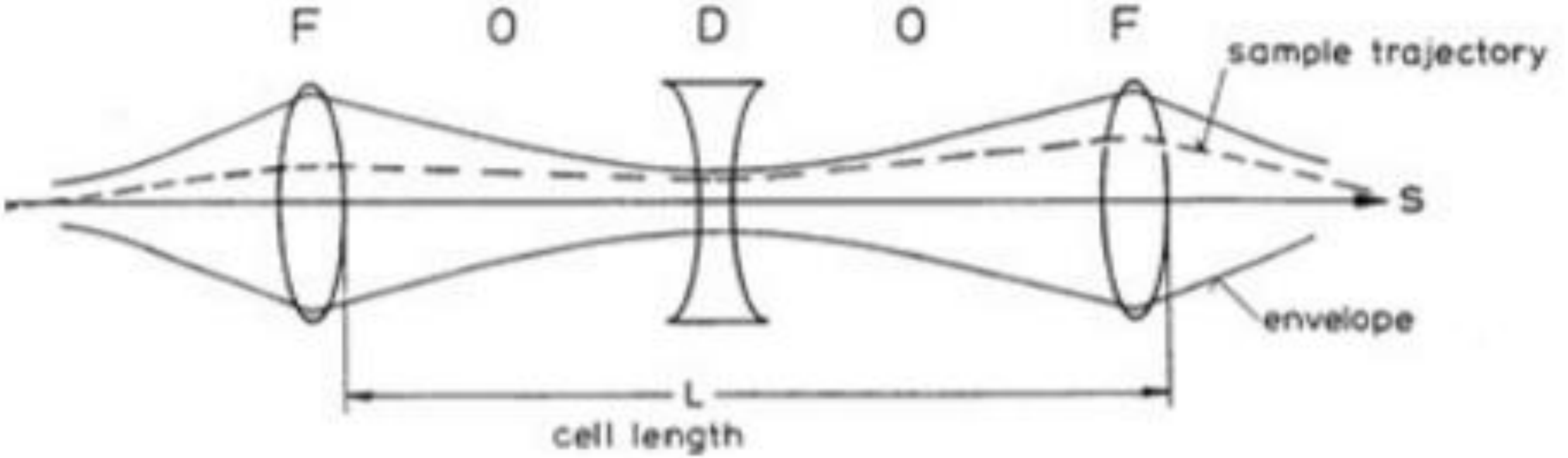
$$\alpha_x'(s) = K(s)\beta_x(s) - \frac{1}{\beta_x(s)}[1 + \alpha_x^2(s)]$$

Amplitude & Beta function

$$\underline{\underline{x(s)}} = \sqrt{2\underline{J_x}\,\underline{\beta_x(s)}}\cos[\underline{\phi_0} + \underline{\Delta\Phi_x(s)}]$$

J_x φ_0 specific to one particle, independent of accelerator location.

β_x $\Delta\Phi_x$ same all particles, depends on accelerator location.

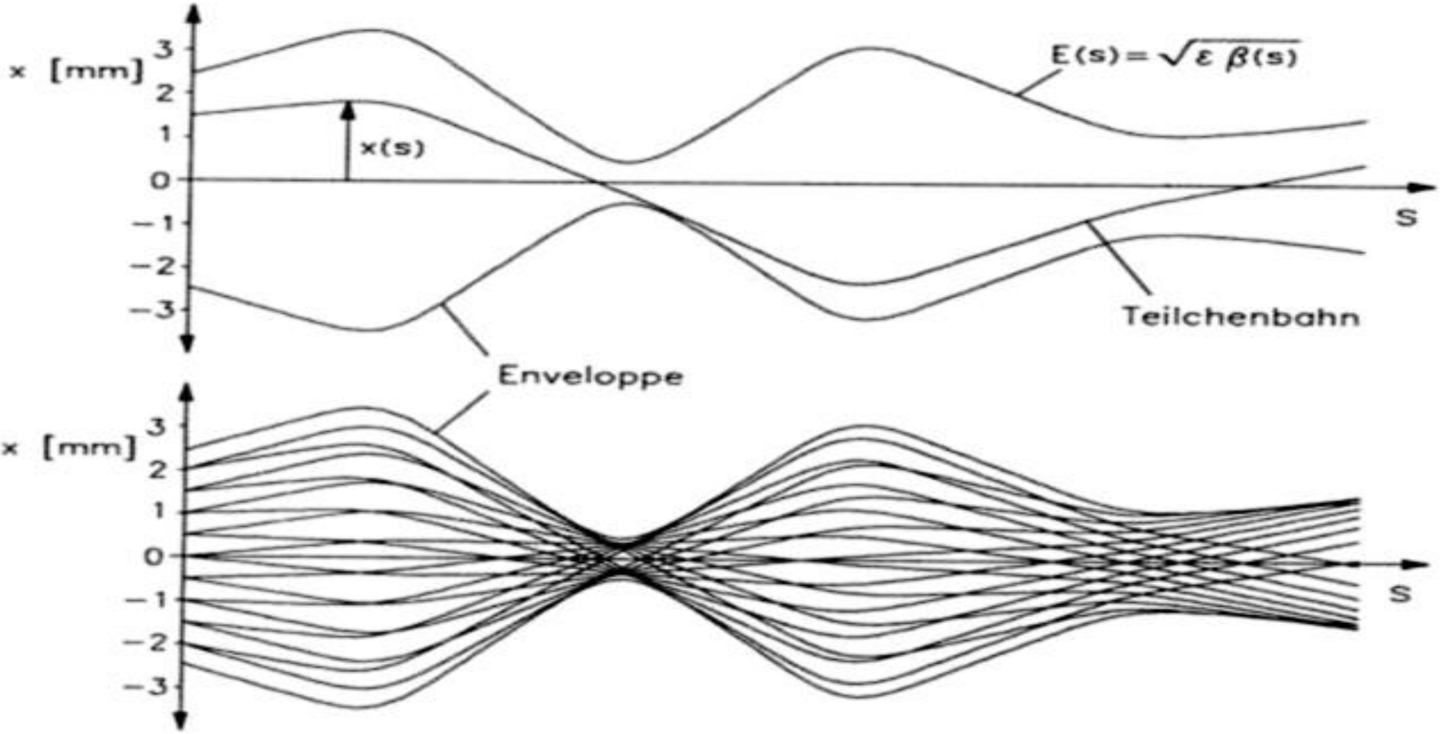


Bartolini

Amplitude & Beta function

$$\underline{\underline{x(s)}} = \sqrt{2\underline{J_x}\underline{\beta_x(s)}}\cos[\underline{\phi_0} + \underline{\Delta\Phi_x(s)}]$$

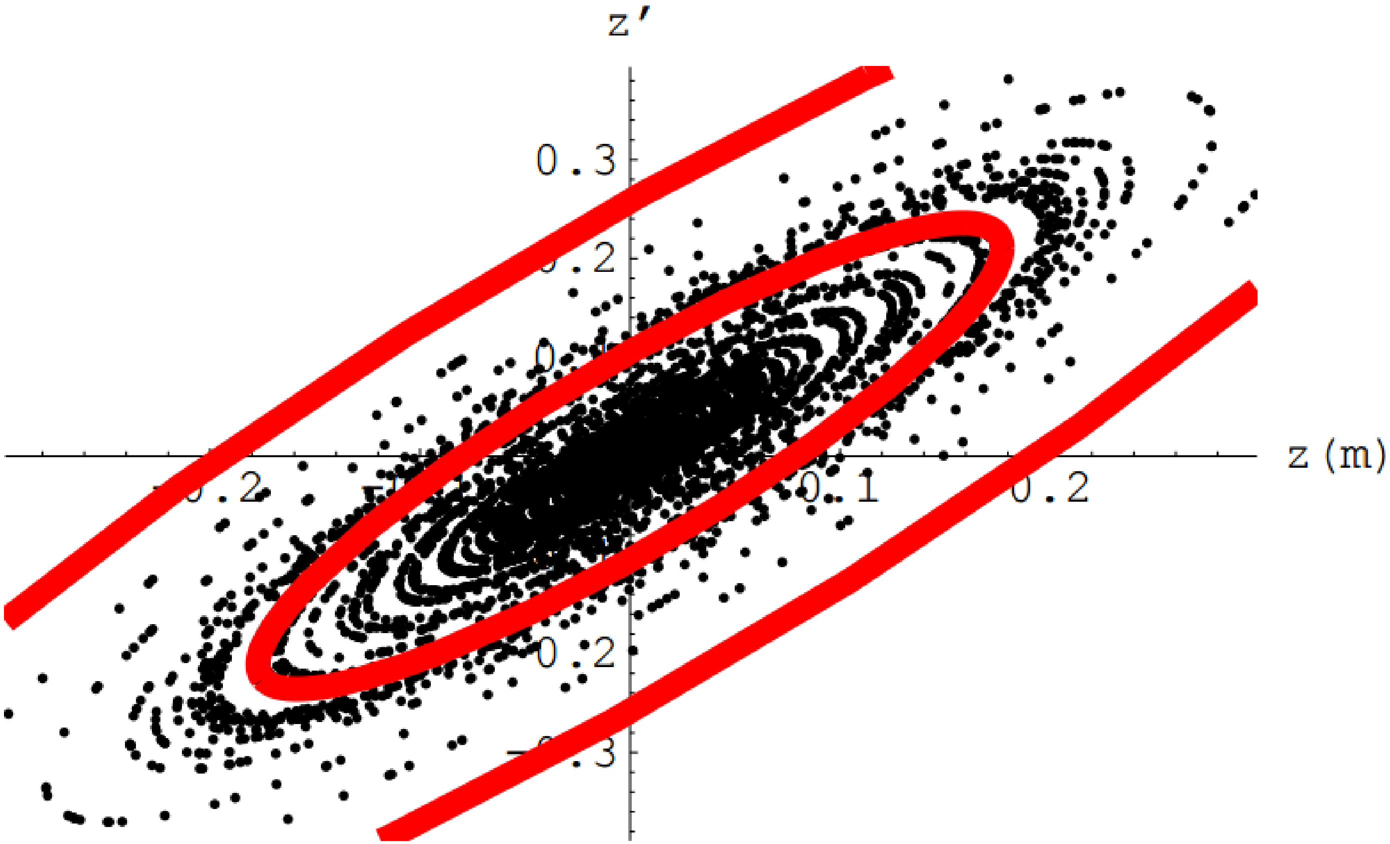
RMS Beam size: $\sigma_x(s) = \sqrt{\epsilon_{rms}\beta_x(s)}$



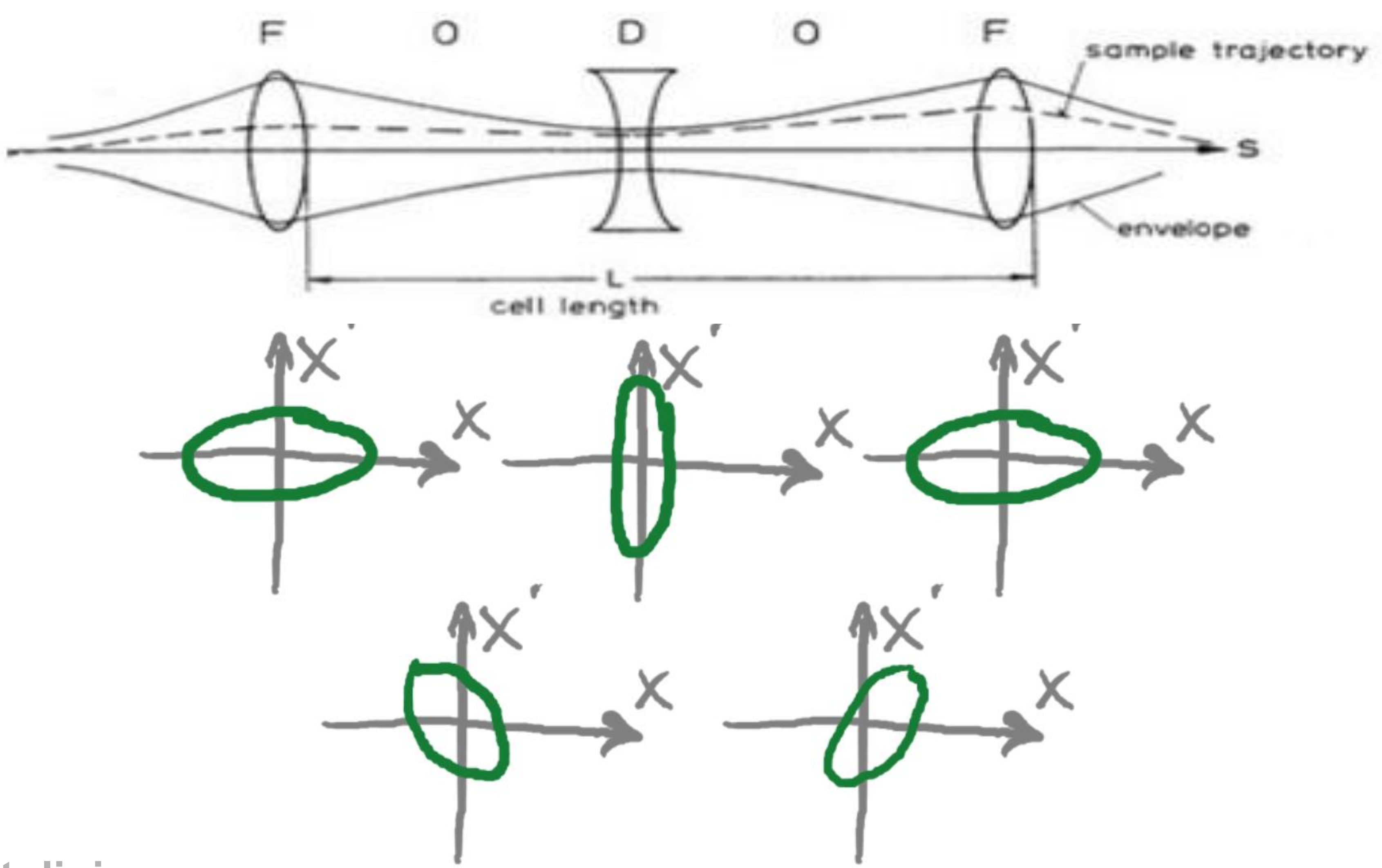
Wille

Transverse Phase-space

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)] \qquad x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\left[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\right]$$



Betatron Motion



Bartolini

Normalized Coordinates

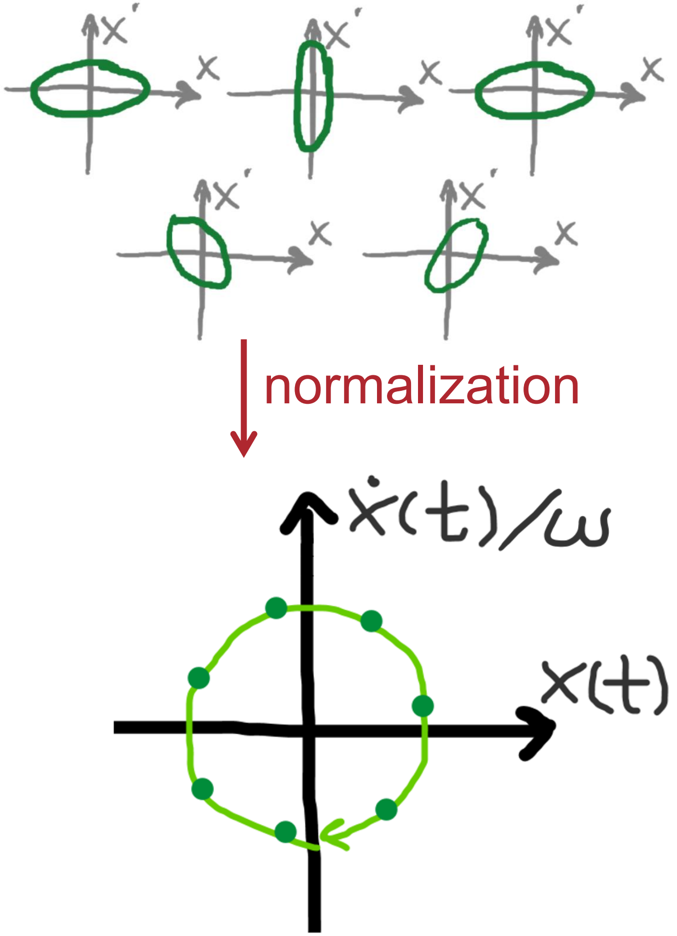
$$x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)] \qquad x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\left[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\right]$$

We can “normalize” these coordinates by a scale-skew transformation:

$$\begin{pmatrix} X \\ P_x \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix}^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$X = \frac{1}{\sqrt{\beta_x}}x = \sqrt{2J_x}\cos[\phi_x(s)]$$

$$P_x = \frac{\alpha_x}{\sqrt{\beta_x}}x + \sqrt{\beta_x}x' = -\sqrt{2J_x}\sin[\phi_x(s)]$$



normalization

Betatron Oscillation

Using these continuous forms of motion:

$$x(s) = \sqrt{2J_x \beta_x(s)} \cos[\phi_x(s)] \qquad x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}} \Big[\sin[\phi_x(s)] + \alpha_x(s) \cos[\phi_x(s)] \Big]$$

Relating this to the matrices, see any general transfer matrix can be parameterized and decomposed:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Phi + \alpha_1 \sin\Delta\Phi) & \sqrt{\beta_1\beta_2}\sin\Delta\Phi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\Phi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\Phi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Phi - \alpha_2\sin\Delta\Phi) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\Delta\Phi & \sin\Delta\Phi \\ -\sin\Delta\Phi & \cos\Delta\Phi \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix}^{-1}$$

An inverse transformation, a rotation, and transformation.

Courant-Snyder (TWISS) Parameters

The transfer matrix for a general transfer matrix:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Phi + \alpha_1 \sin\Delta\Phi) & \sqrt{\beta_1\beta_2}\sin\Delta\Phi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\Phi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\Phi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Phi - \alpha_2\sin\Delta\Phi) \end{pmatrix}$$

Transfer matrix for an entire ring, impose $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$:

$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\alpha_x = -\frac{\beta'_x}{2}$$

$$\gamma_x = \frac{1+\alpha_x^2}{\beta_x}$$

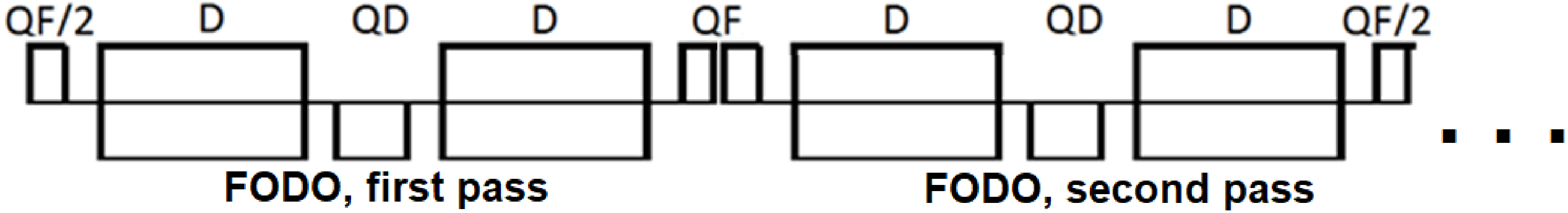
These α, β are known as Courant-Snyder or TWISS parameters. We can think of them either as parameterization of the transfer matrix or as functions which solve the Hill's equation.

Courant-Snyder (TWISS) Parameters

The transfer matrix for an entire ring

$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

For example, we can calculate TWISS for a repeating FODO ring:

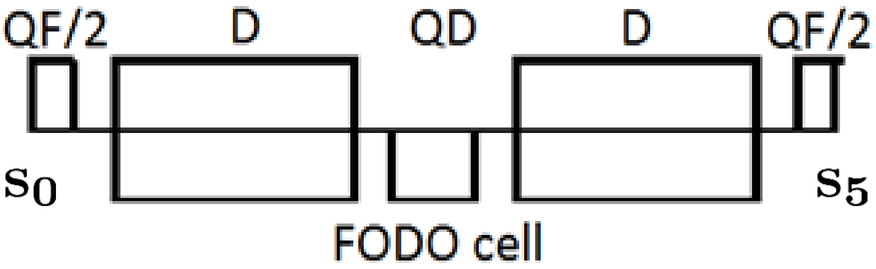


$$M(s_0) = \begin{pmatrix} 1 - \frac{L^2}{2f} & 2L\left(1 + \frac{L}{2f}\right) \\ -\frac{L}{2f^2}\left(1 - \frac{L}{2f}\right) & 1 - \frac{L^2}{2f} \end{pmatrix}$$

$$\cos\Phi = \frac{1}{2}\mathrm{Tr}(M) = 1 - L^2/2f$$

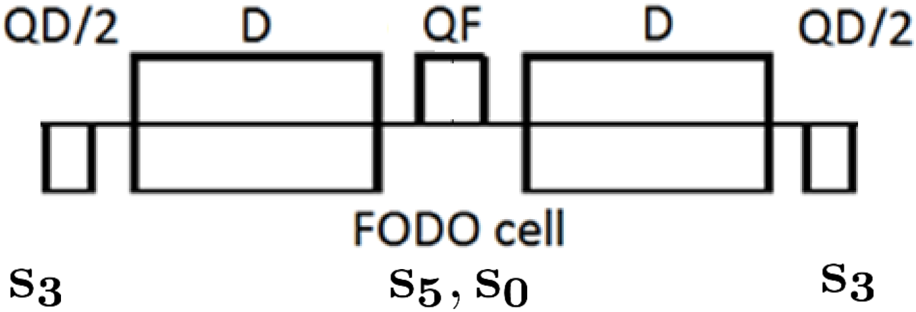
$$\beta = \frac{(2L)(1 + L/2f)}{\sin\Phi} \qquad \alpha = 0$$

Question – Where is this the beta function for?



$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$



$$M = \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix}$$

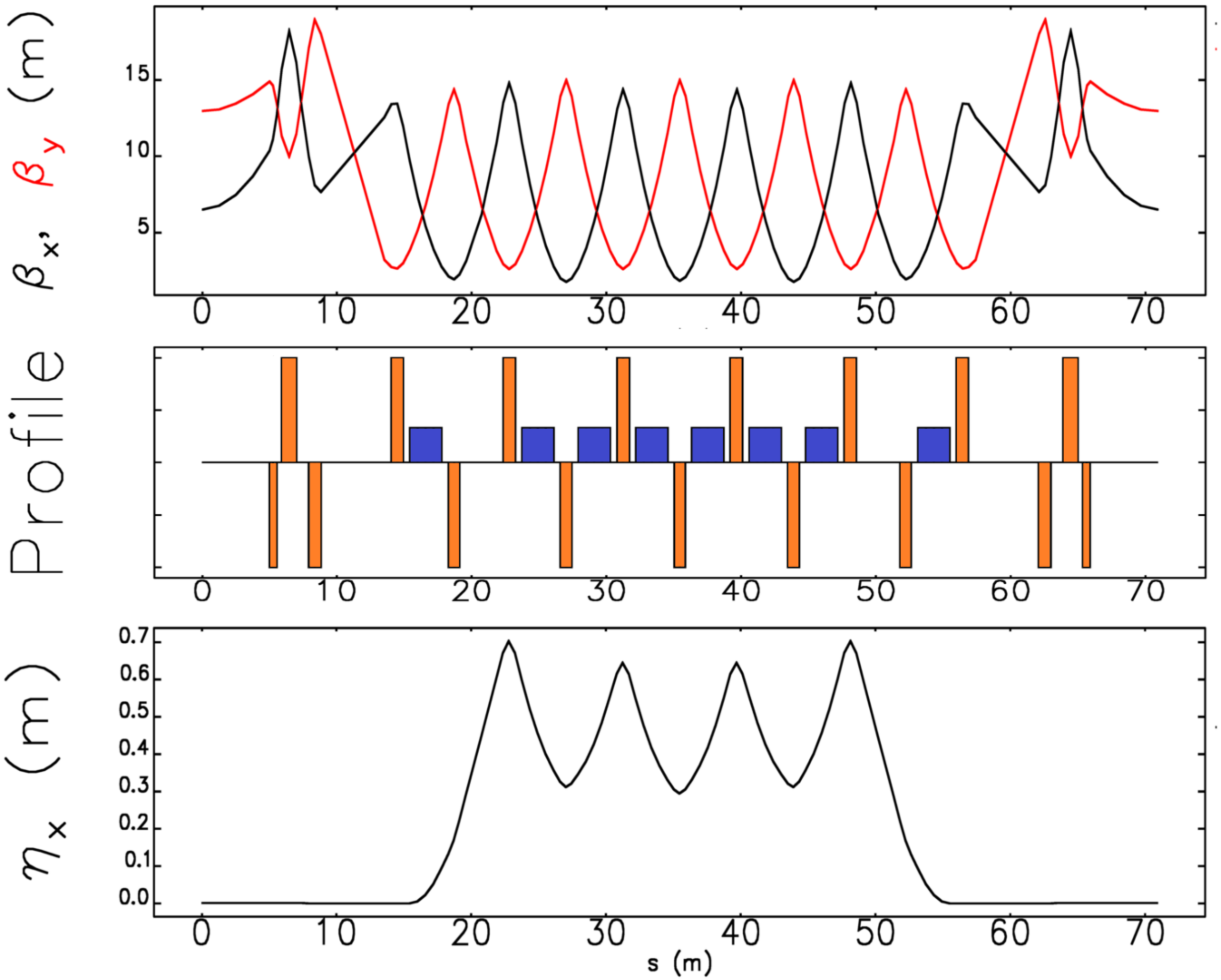
$$M = \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 - \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 + \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$

Matrix order: $M(s_0) = M(s_0|s_5) \cdot M(s_5|s_4) \cdot M(s_4|s_3) \cdot M(s_3|s_2) \cdot M(s_2|s_1) \cdot M(s_1|s_0)$

vs. $M(s_3) = M(s_3|s_2) \cdot M(s_2|s_1) \cdot M(s_1|s_0) \cdot M(s_0|s_5) \cdot M(s_5|s_4) \cdot M(s_4|s_3)$

Similarity transform: $M(s_3) = [M(s_3|s_0)] \cdot M(s_0) \cdot [M(s_3|s_0)]^{-1}$

Computed Calculation of TWISS Plots



Stability over many revolutons

The ring transfer matrix at location s_1:

$$M(s_1) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos(2\pi\nu_x) & \sin(2\pi\nu_x) \\ -\sin(2\pi\nu_x) & \cos(2\pi\nu_x) \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}^{-1}$$

This is the “one-turn map”, the effect on the beam as starts at s_1, travels around the ring, and returns to s_1.

The effect of N turns:

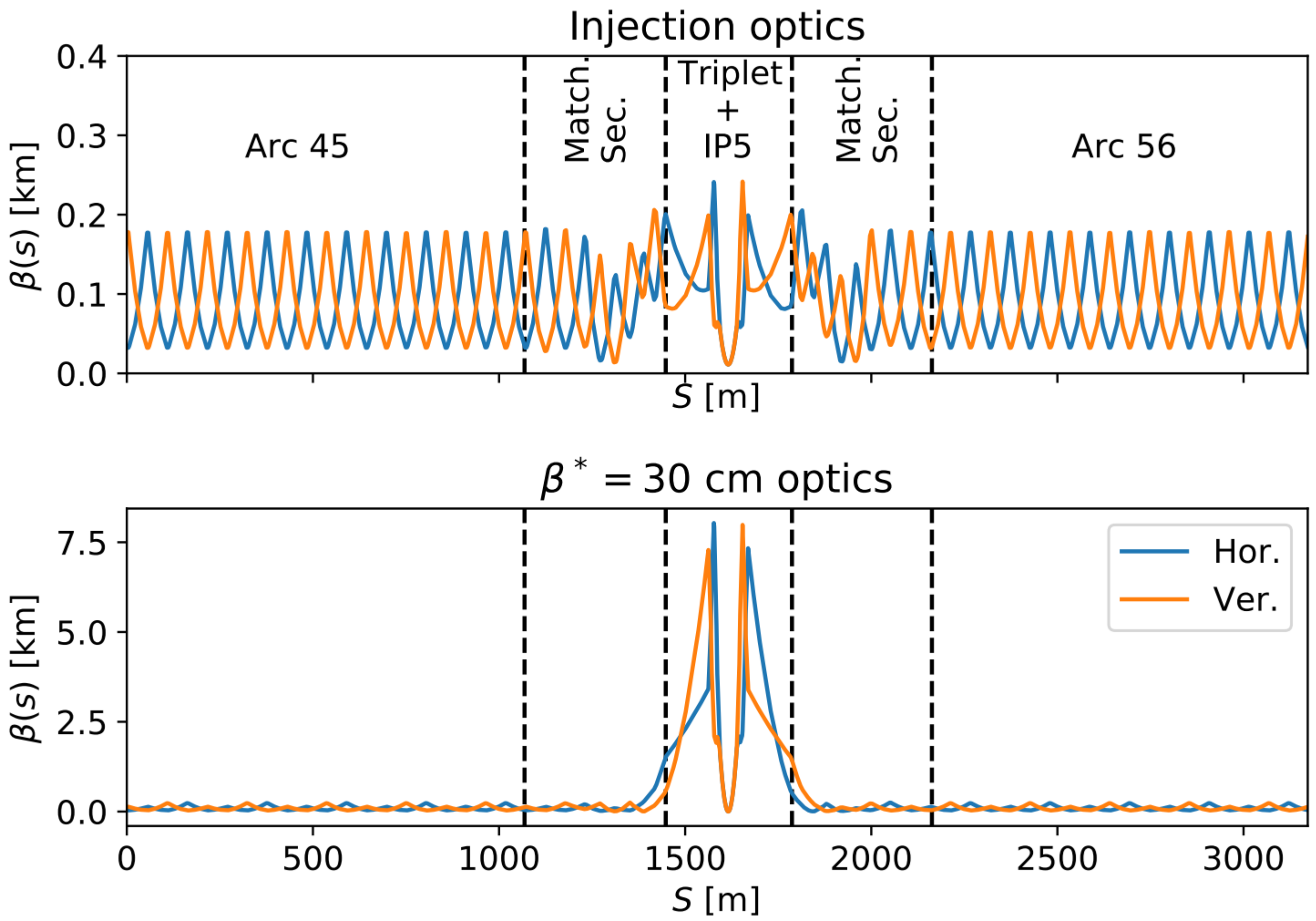
$$M(s_1)^N = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos(2\pi N\nu_x) & \sin(2\pi N\nu_x) \\ -\sin(2\pi N\nu_x) & \cos(2\pi N\nu_x) \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}^{-1}$$

The eigenvalues of M are: $e^{i\nu_x}, \quad e^{-i\nu_x}$

they must be unimodular reciprocals.

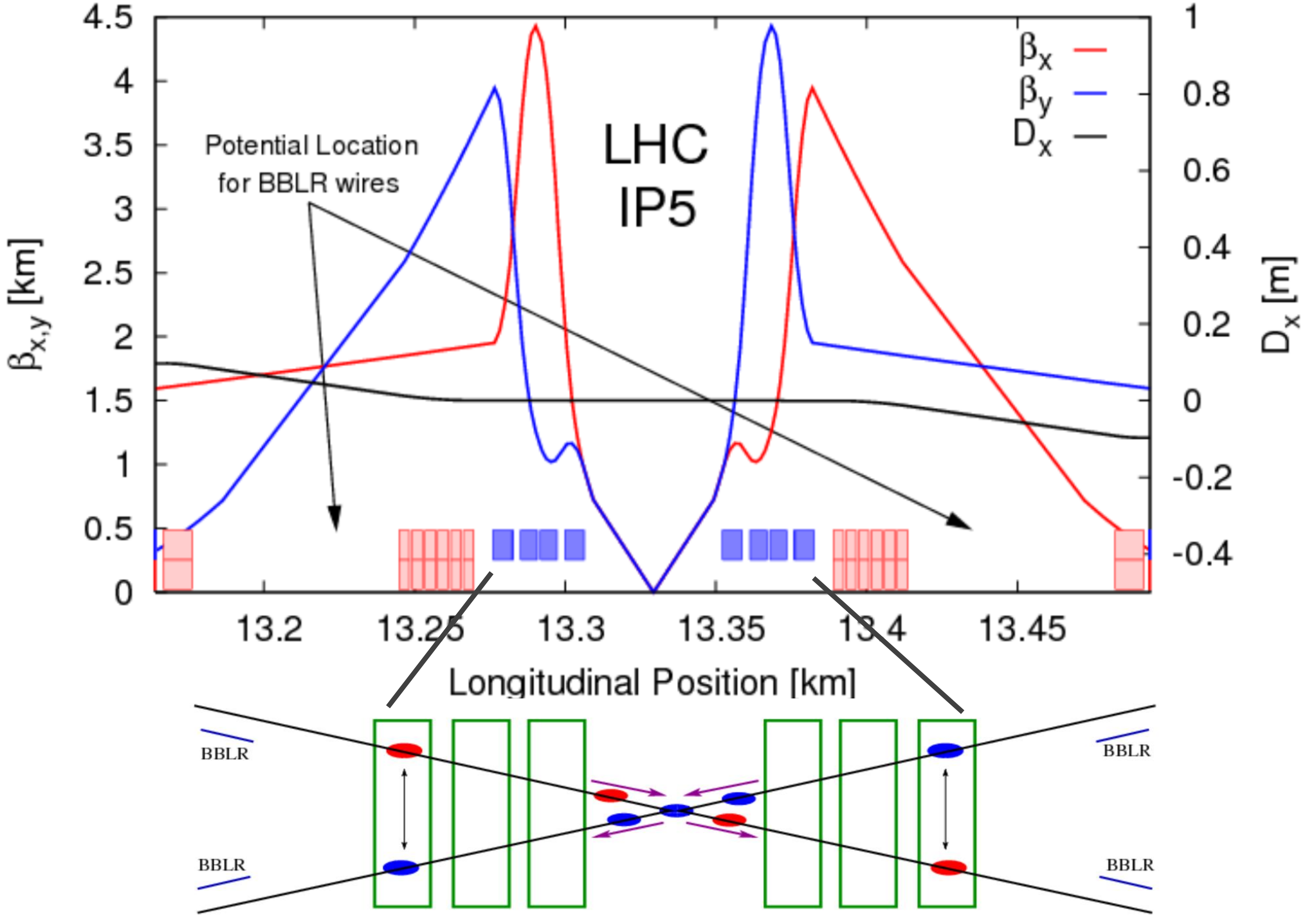
ν_x or Q_x, the betatron tune.

LHC IP5 TWISS Plot



R Calaga, Long-range beam-beam experiments in the relativistic heavy ion collider

LHC IP5 TWISS Plot (cont.)



Horz: Q_D Q_F Q_D Q_D Q_F Q_D

Vert: Q_F Q_D Q_F Q_F Q_D Q_F

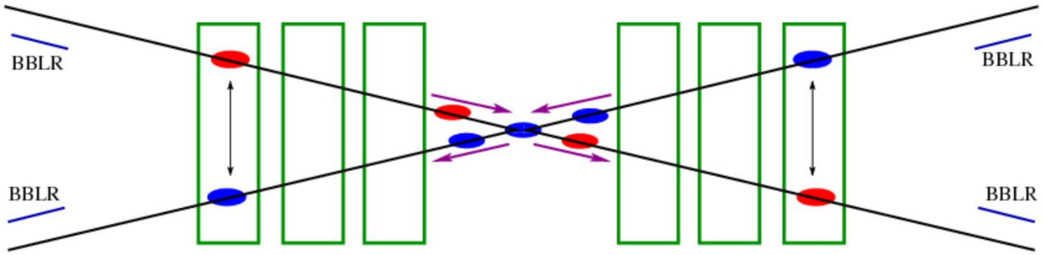
Jaime Maria Coello, Dissertation

Collider Optics

In any drift section, the beta function is given by:

$$\beta(s) = \beta^* \left[1 + \left(\frac{s - s^*}{\beta^*}\right)^2\right]$$

where β* is the minimum beta, and s* is the location of that minimum. Interaction points (IPs) are typically placed in drifts with the beam tightly focused to a minimum value in both planes and from a large beta value on both sides.

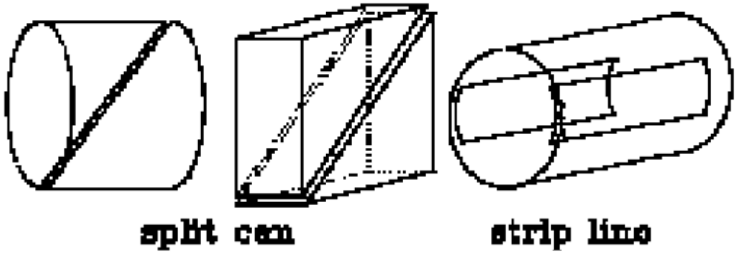


Horz: Q_D Q_F Q_D Q_D Q_F Q_D
Vert: Q_F Q_D Q_F Q_F Q_D Q_F

Collimators and Phase-Advance

Two Beam Position Monitor Measurement

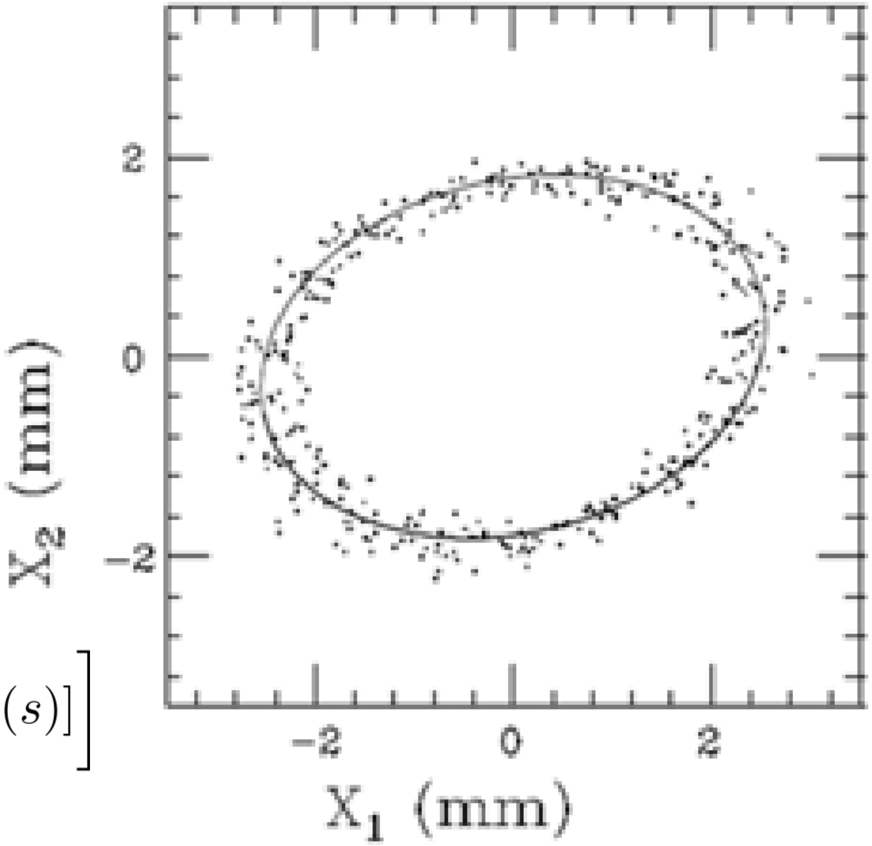
The correlation between X_1 and X_2, can be used to see the betatron phase advance between those points.



$$y \approx \frac{w}{2}\frac{U_+ - U_-}{U_+ + U_-} = \frac{w}{2}\frac{\Delta}{\Sigma},$$

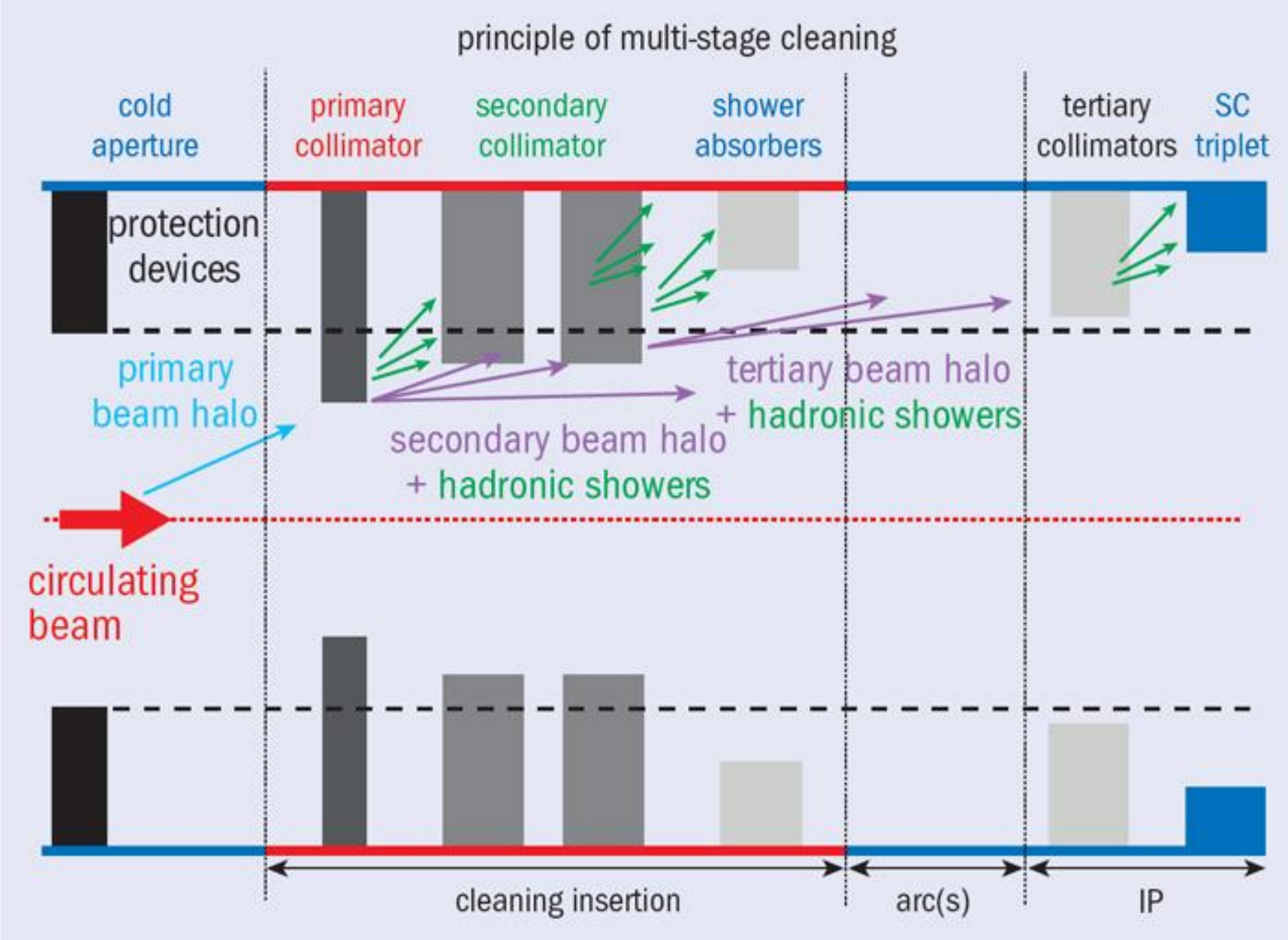
$$x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)]$$

$$x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\left[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\right]$$



SY Lee

Machine Protection through Collimation



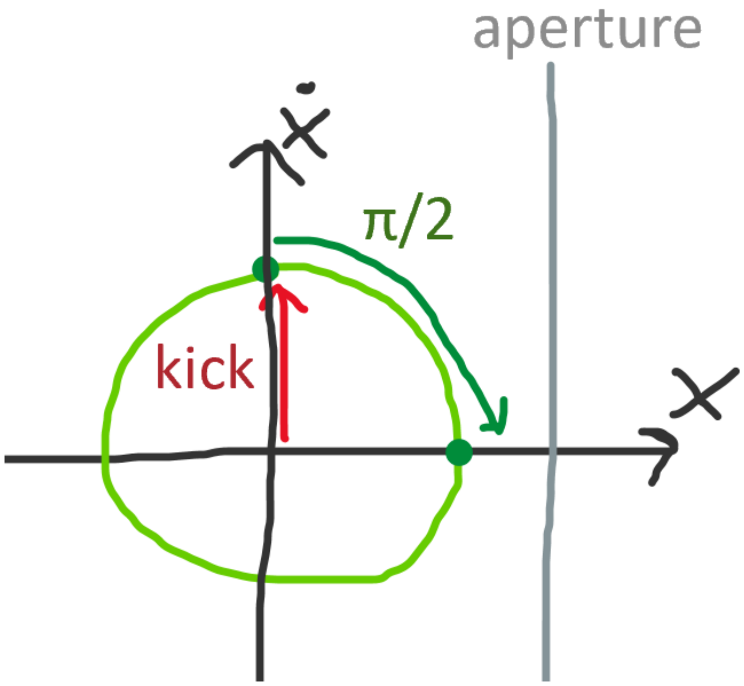
CERN Courier

Betatron Phase Advance

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x)$$

$$x' = -\sqrt{\frac{2J_x}{\beta_x}}[\sin(\phi_x) + \alpha_x \cos(\phi_x)]$$

$$\phi_x(s_2) - \phi_x(s_1) = \Delta\phi_x = \int_{s_1}^{s_2} \frac{1}{\beta_x(s)} ds'$$



Kick amplitude after π/2:

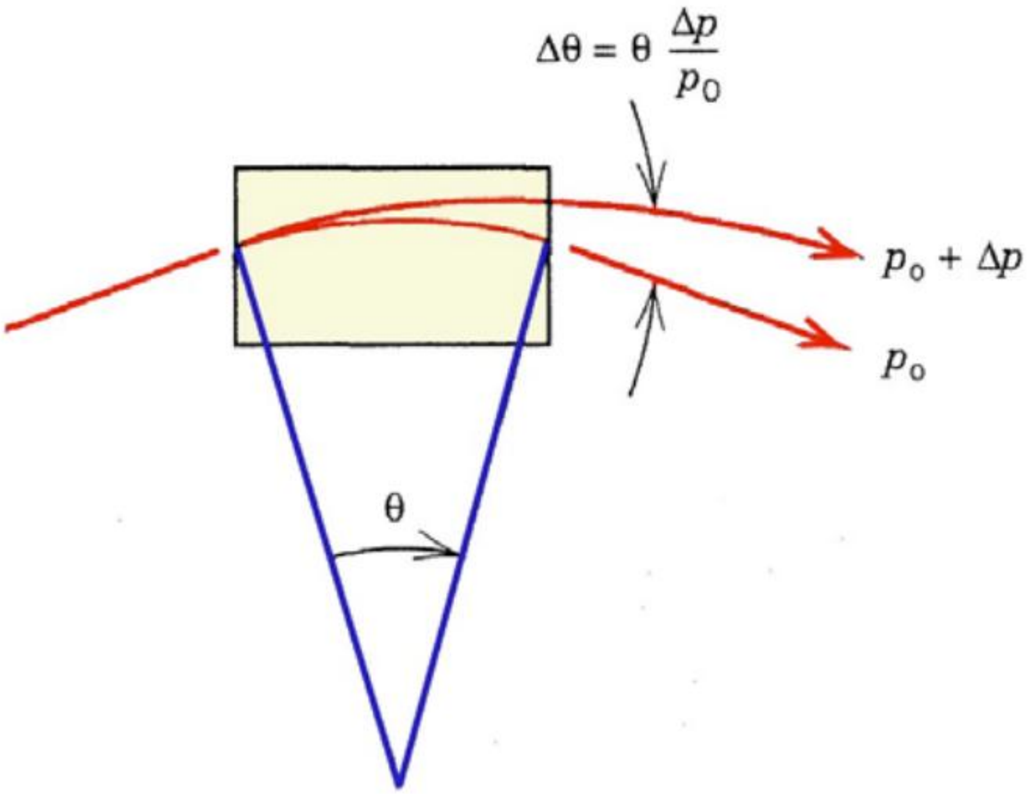
$$x_1' = \Delta\theta = \sqrt{\frac{2J_x}{\beta_{x1}}}$$

$$J_x = \frac{1}{2}\beta_{x1}\Delta\theta^2$$

$$x_2 = \sqrt{2\beta_{x2}J_x} = \sqrt{\beta_{x1}\beta_{x2}}\Delta\theta$$

Off-Momentum Particles & Sextupole Magnets

Dispersion



$$\delta \equiv \frac{p - p_0}{p_0}$$

Dispersion:

$$D'' + K_x(s)D = \frac{1}{\rho}$$

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x) + D\delta$$

Spot Size:

$$\sigma_{x,rms}^2 = \beta_x \epsilon_{rms} + D^2\delta^2$$

Barletta

Chromaticity

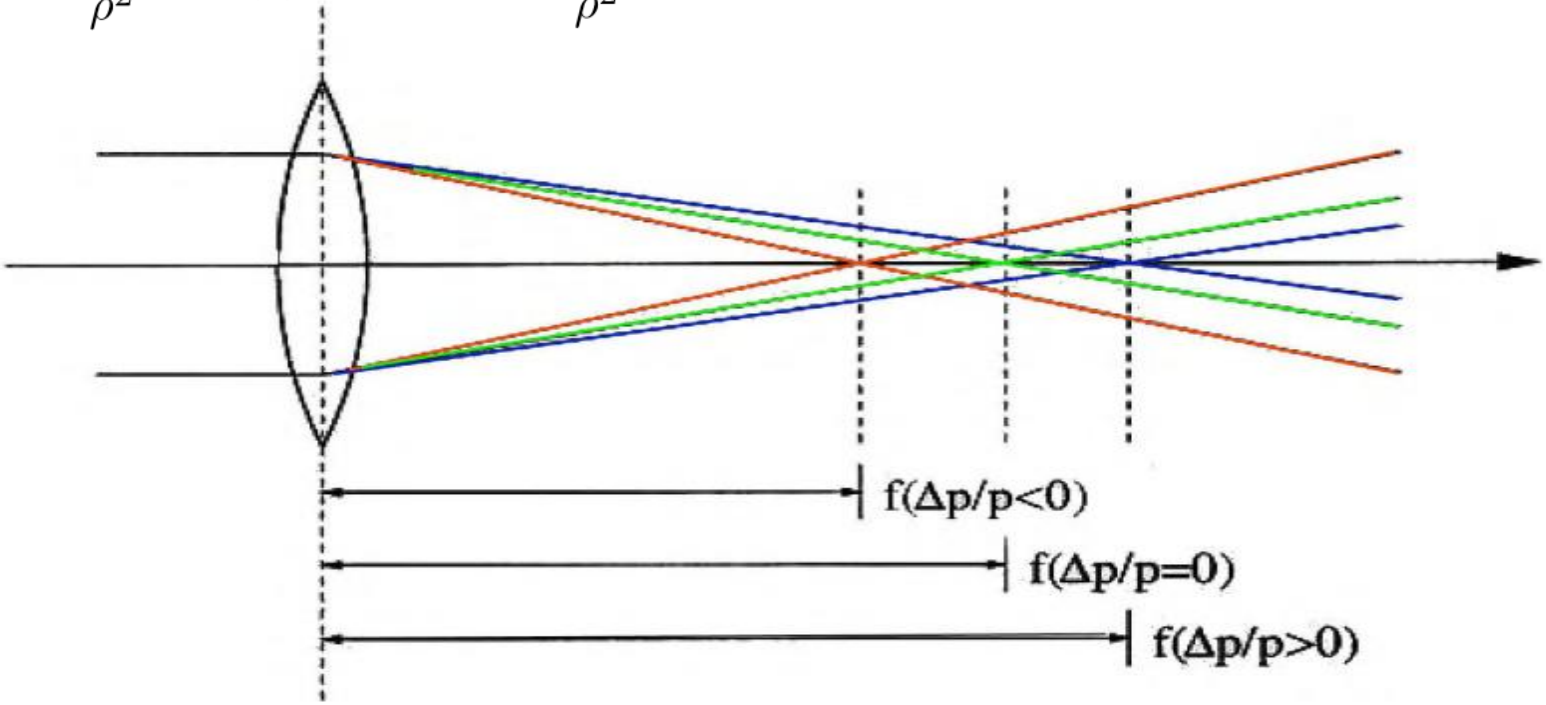
Change in tune with momentum:

$$x''_\beta + (K_x + \Delta K_x \delta) x_\beta = 0$$

$$K_x = \frac{1}{\rho^2} - K(s) \qquad \Delta K_x = -\frac{2}{\rho^2} + K(s)$$

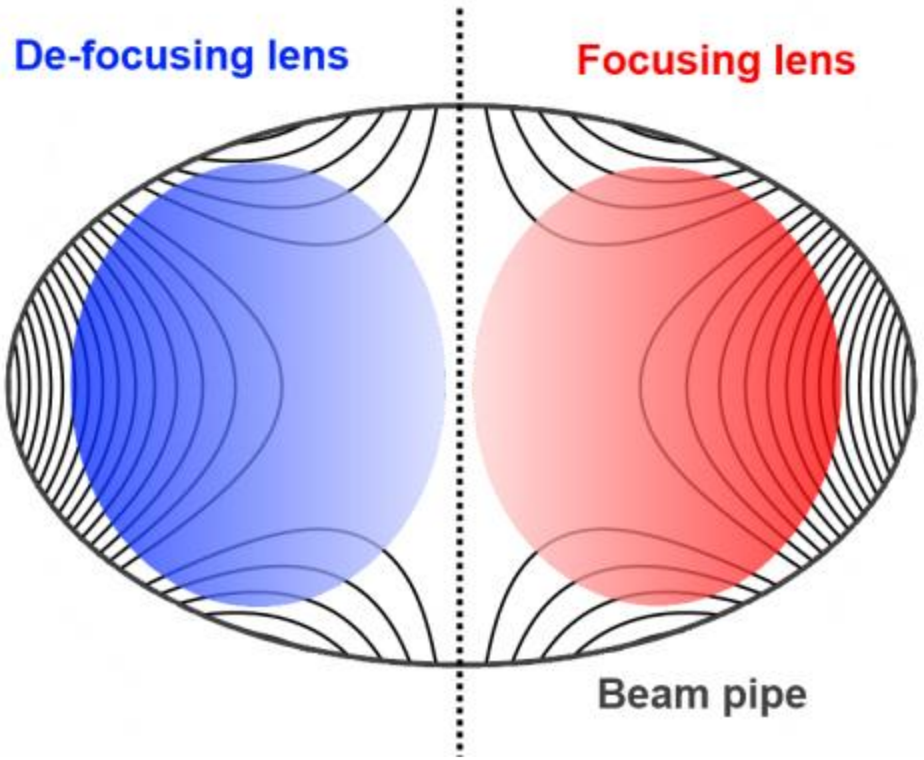
Chromaticity:

$$\frac{\partial Q_{x,y}}{\partial \delta} = Q'_{x,y} = \frac{1}{4\pi} \int_0^C \beta_{x,y} \Delta K_{x,y}(s) ds$$



Barletta

Sextupoles & Chromaticity Correction



Dispersion is position offset dependence on momentum.

Chromaticity is tune dependence on momentum.

Sextupoles provide tune-shift depending on position offset.

$$Q'_{x,y} = \frac{1}{4\pi} \int_0^C \beta_{x,y} \left[\Delta K_{x,y}(s) + S(s) D_x(S) \right] ds$$

$$S(s) = \frac{\partial^2 B_z(s)}{\partial x^2} \frac{1}{B\rho}$$

FNAL Rookie Book

Average Lattice Parameters

For a typical FODO-like lattice we can estimate some parameters

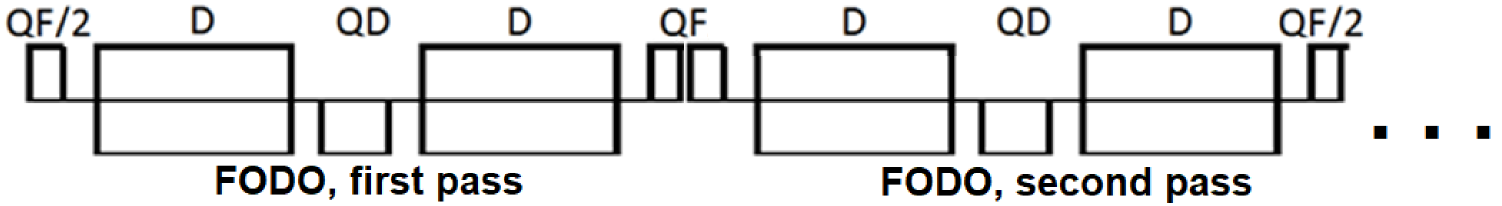
Natural Chromaticity: $Q'_{x,y} \approx -Q_x$

Average Betas: $\beta_{xy,ave} \approx R/Q_{x,y}$

Average Dispersion: $D_{x,ave} \approx R/Q_x^2$

Transition gamma: $\gamma_T \approx Q_x$

where $R = C/(2\pi)$



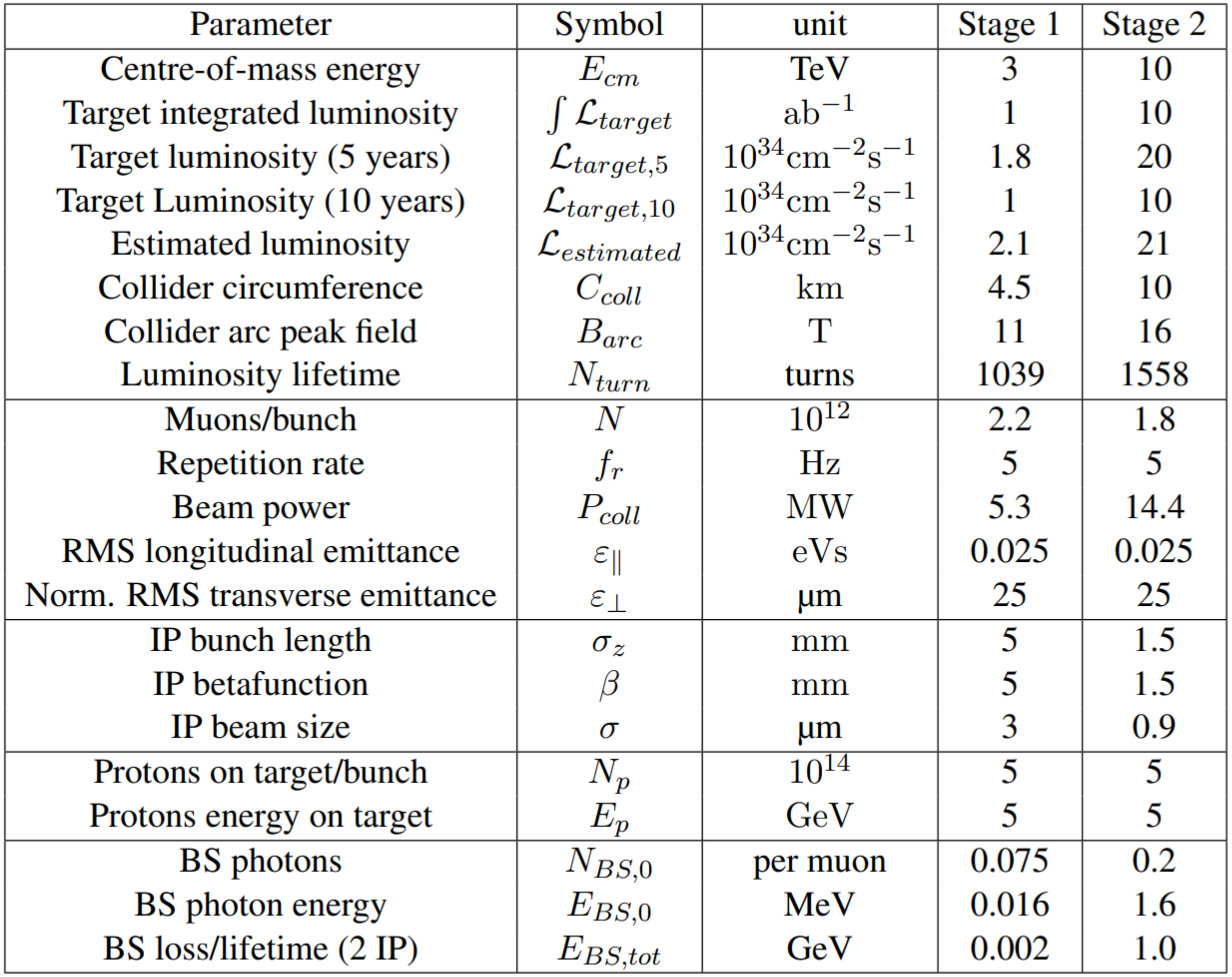
$$M(s_0) = \begin{pmatrix} 1 - \frac{L^2}{2f} & 2L\left(1 + \frac{L}{2f}\right) \\ -\frac{L}{2f^2}\left(1 - \frac{L}{2f}\right) & 1 - \frac{L^2}{2f} \end{pmatrix}$$

$$\cos\Phi = \frac{1}{2}\mathrm{Tr}(M) = 1 - L^2/2f$$

$$\beta_{x,y} = \frac{2L(1 \pm L/2f)}{\sin\Phi}$$

IMCC EU Muon Collider Parameters

Full list of parameters here (2023 EU): here



Parameter	Symbol	unit	Stage 1	Stage 2
Centre-of-mass energy	E_{cm}	TeV	3	10
Target integrated luminosity	$\int \mathcal{L}_{target}$	ab^{-1}	1	10
Target luminosity (5 years)	$\mathcal{L}_{target,5}$	$10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$	1.8	20
Target Luminosity (10 years)	$\mathcal{L}_{target,10}$	$10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$	1	10
Estimated luminosity	$\mathcal{L}_{estimated}$	$10^{34}\mathrm{cm}^{-2}\mathrm{s}^{-1}$	2.1	21
Collider circumference	C_{coll}	km	4.5	10
Collider arc peak field	B_{arc}	T	11	16
Luminosity lifetime	N_{turn}	turns	1039	1558
Muons/bunch	N	10^{12}	2.2	1.8
Repetition rate	f_r	Hz	5	5
Beam power	P_{coll}	MW	5.3	14.4
RMS longitudinal emittance	$\varepsilon_{\parallel}$	eVs	0.025	0.025
Norm. RMS transverse emittance	$\varepsilon_{\perp}$	µm	25	25
IP bunch length	σ_z	mm	5	1.5
IP betafunction	β	mm	5	1.5
IP beam size	σ	µm	3	0.9
Protons on target/bunch	N_p	10^{14}	5	5
Protons energy on target	E_p	GeV	5	5
BS photons	$N_{BS,0}$	per muon	0.075	0.2
BS photon energy	$E_{BS,0}$	MeV	0.016	1.6
BS loss/lifetime (2 IP)	$E_{BS,tot}$	GeV	0.002	1.0

Summary

The linear transverse dynamics of a particle accelerator are governed by Hill's Equation, which is a time-varying harmonic oscillator.

We calculate the trajectory of individual particles through the many individual magnets of a particle accelerator using transfer matrices.

Transfer matrices are also used for the beam size and oscillation phase, which are represented by Courant-Snyder parameters.

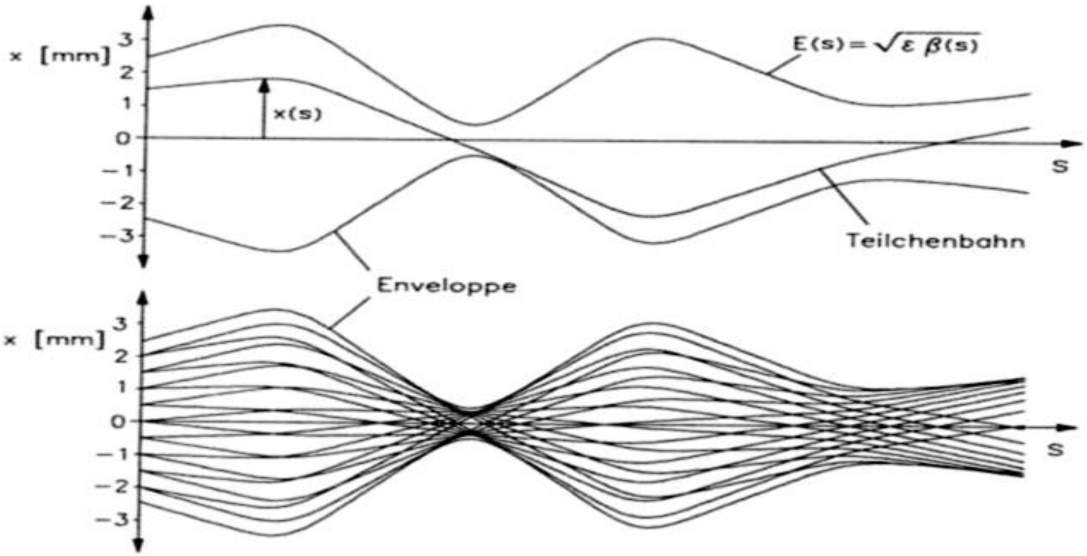
There are chromatic effects, resonances, and space-charge effects that complicate the process of designing and operating a particle accelerator.

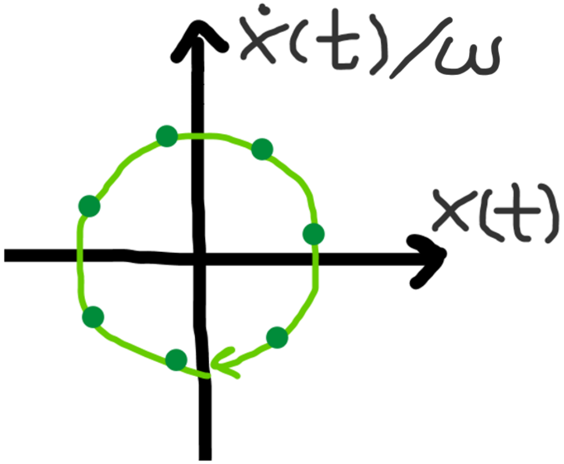
Some backup slides on longitudinal dynamics.

Betatron Resonances

Discrete Sampling

Depending on the ratio between betatron frequency the revolution frequency, the phase of oscillation with each passage of the beam may fall under regular patterns.



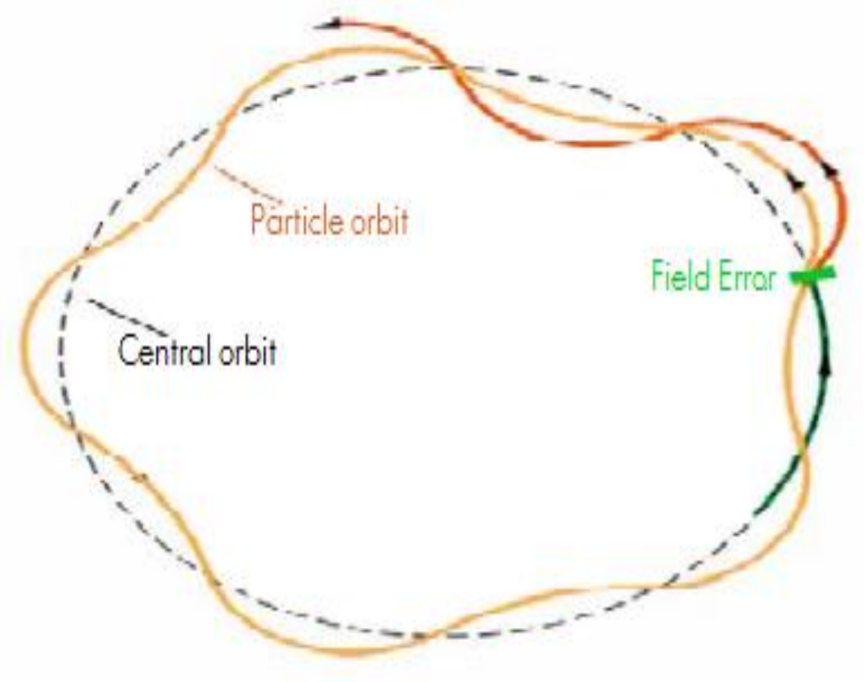


Betatron Tune: $\nu = \frac{1}{2\pi}\int_{s_0}^{s_0+C} \frac{1}{\beta_x(s)} ds'$

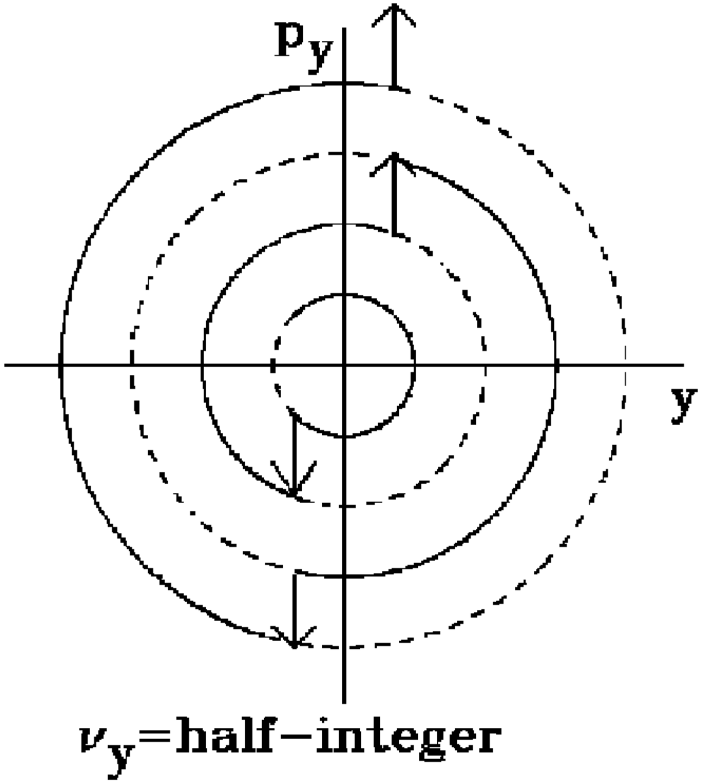
Betatron Tune Resonance

Perturbation will accumulate if tune is a fraction corresponding to the symmetry of the applied fields.

Dipole-Integer Resonace:

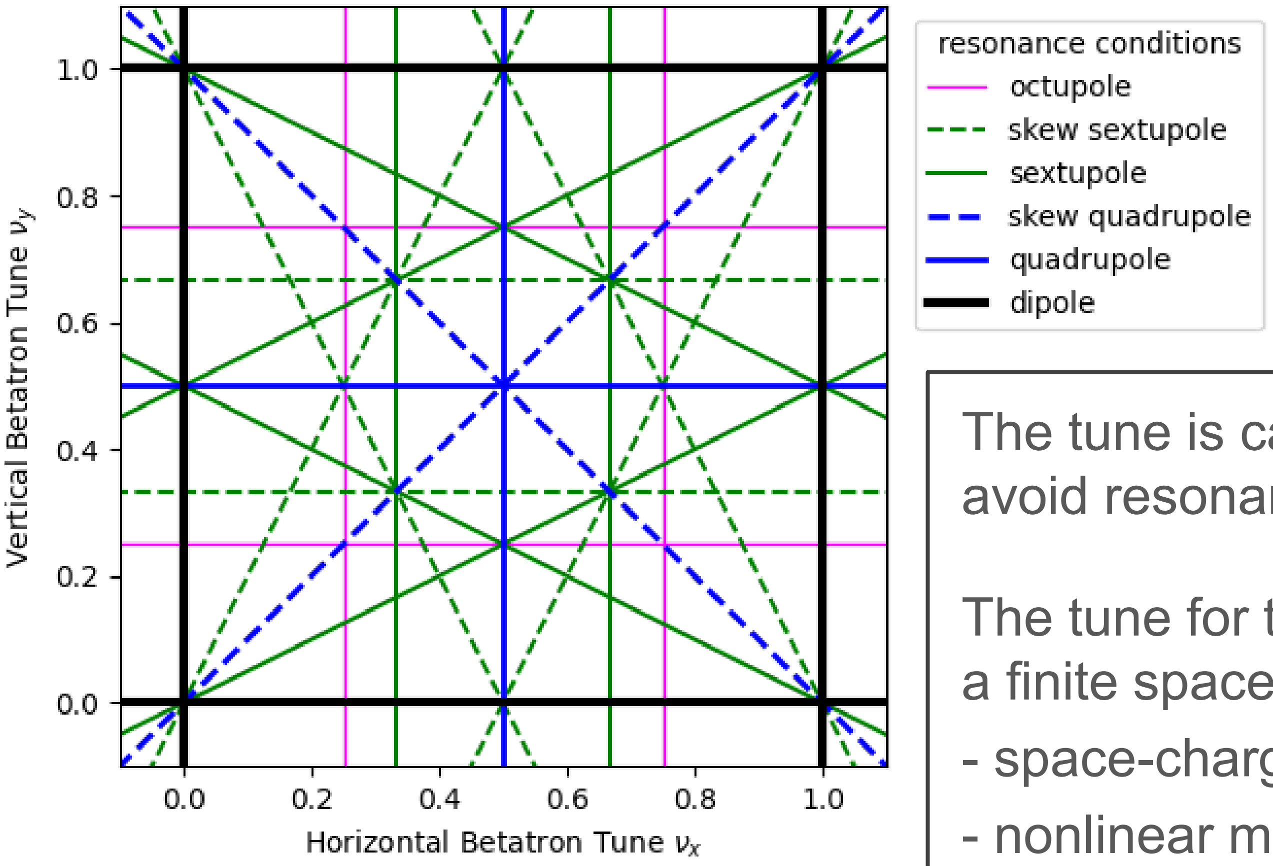


Quadrupole Resonace:



SY Lee

Tune Diagrams (by magnet-type)



The tune is carefully picked to avoid resonances.

The tune for the beam occupies a finite space:

- space-charge tune spread.
- nonlinear magnets.
- chromaticity.

Every Magnet Contributes to the Resonance

Examples of resonance calculations:

$$G_{dipole} \propto \int \left[\theta_{x,y}(s)\beta_{x,y}^{1,2}(s)e^{i\phi_{x,y}(s)}\right] ds$$

$$G_{quad} \propto \int \left[K_{x,y}(s)\beta_{x,y}(s)e^{2i\phi_{x,y}(s)}\right] ds$$

$$G_{3,0} \propto \int \left[S_x(s)\beta_x^{3/2}(s)e^{3i\phi_x(s)}\right] ds$$

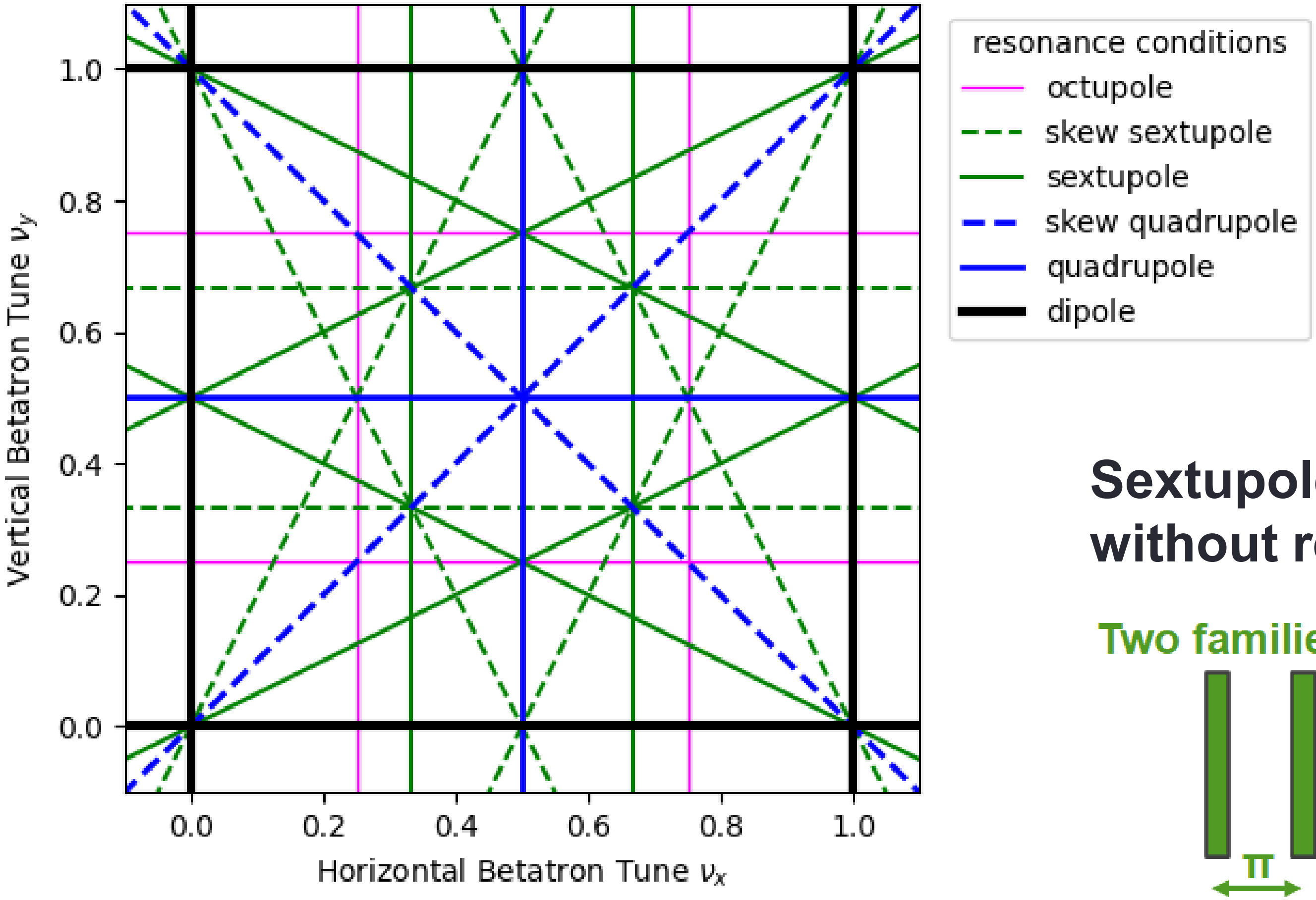
$$G_{1,\pm2} \propto \int \left[S_x(s)\beta_x^{1/2}(s)\beta_y(s)e^{i\phi_x(s)\pm 2i\phi_y(s)}\right] ds$$

The magnetic multipole, multiplied by a corresponding amplitude and phase.

Generically, there will be many resonances and the lowest order resonances will be the strongest.

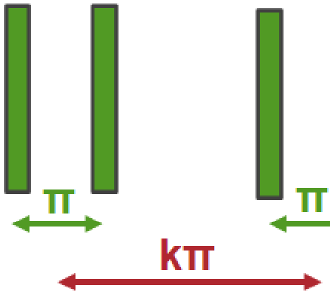
However, by carefully designing the overall accelerator structure with symmetries, many resonances can be cancelled simultaneously.

Tune Diagrams (by magnet-type)

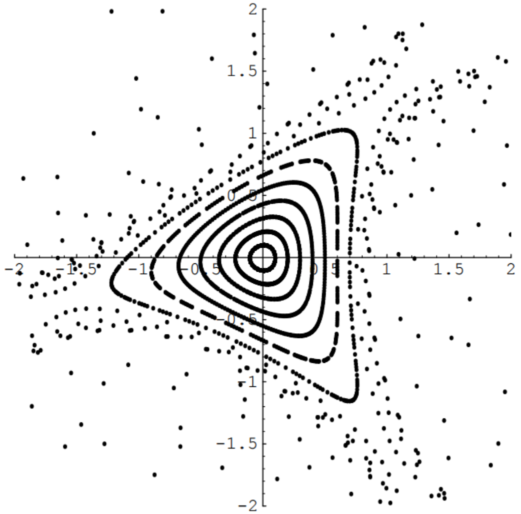


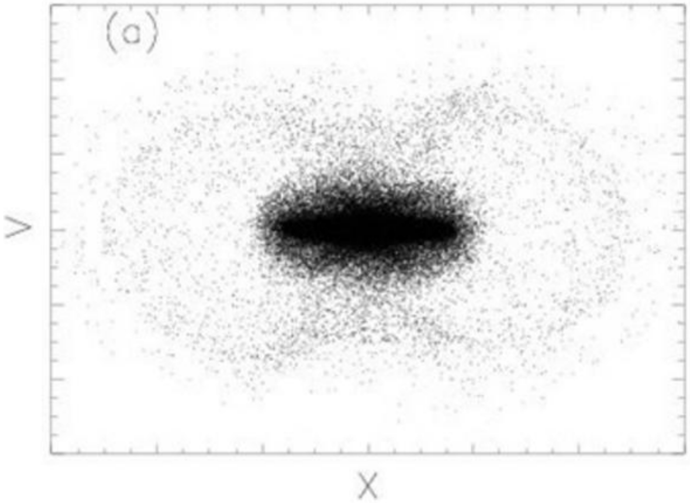
Sextupole configuration without resonances

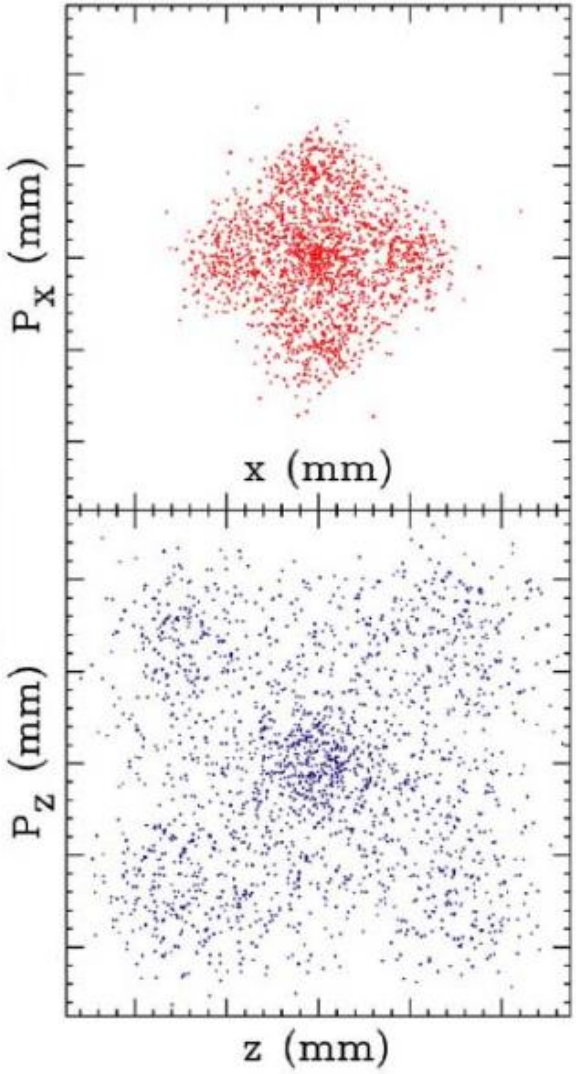
Two families of sextupoles:

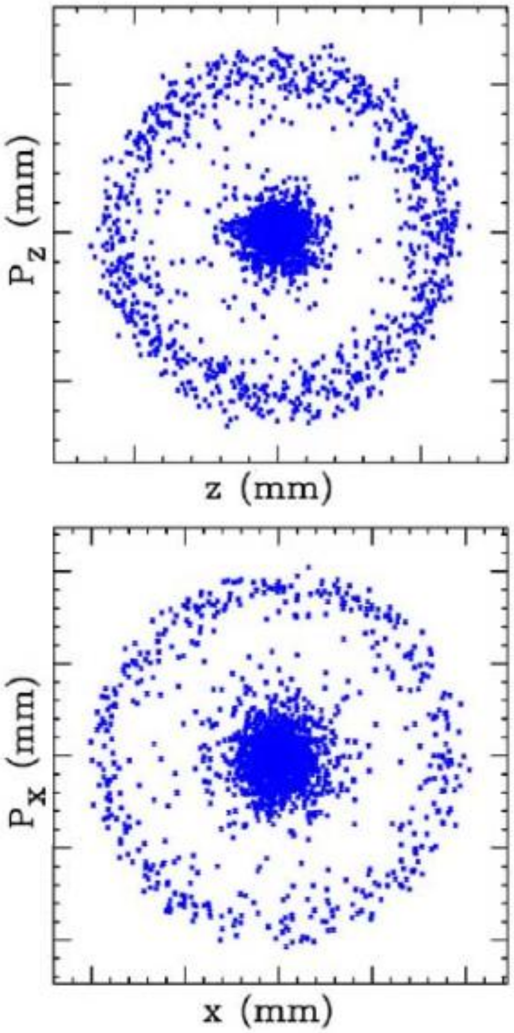


Phase-space Distortions

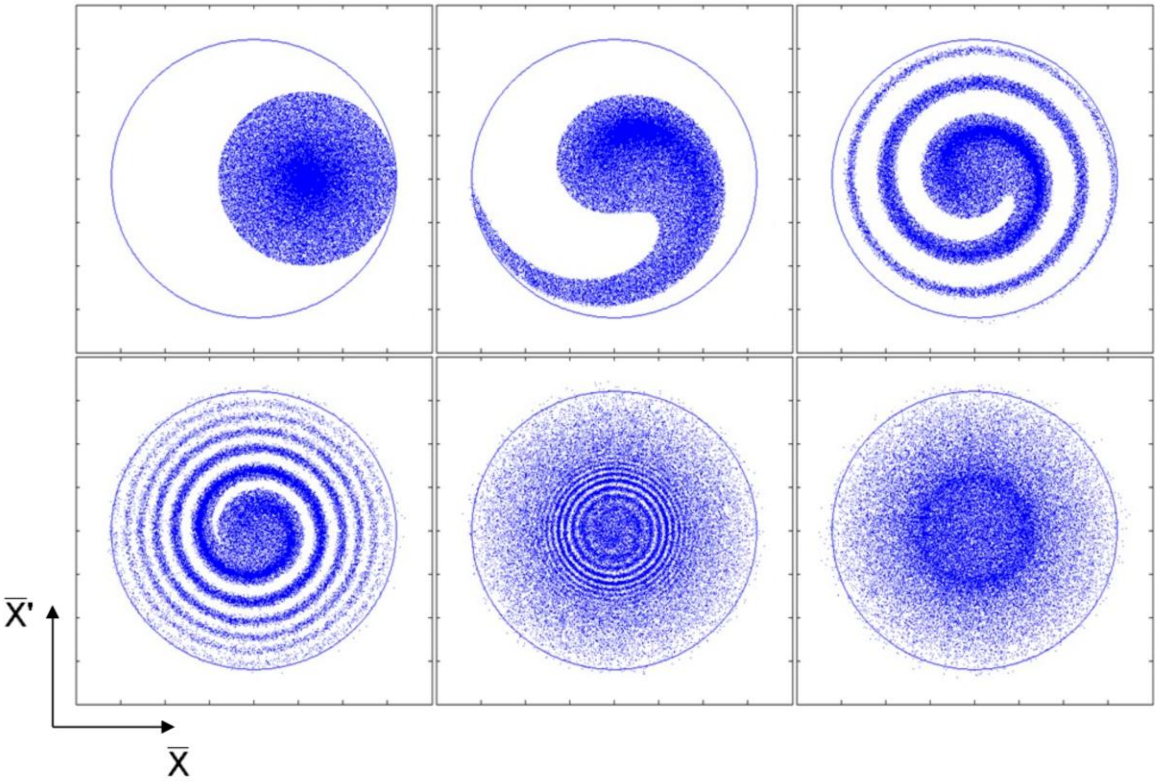








Nonlinear Decoherence



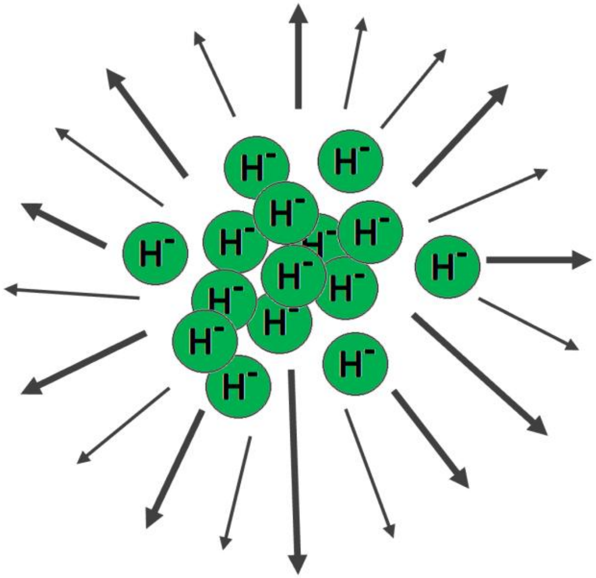
Injection errors, instabilities, and sudden lattice changes may cause a phase-space mismatch.

Tune spread causes the beam to fill-out along the phase-space contours.

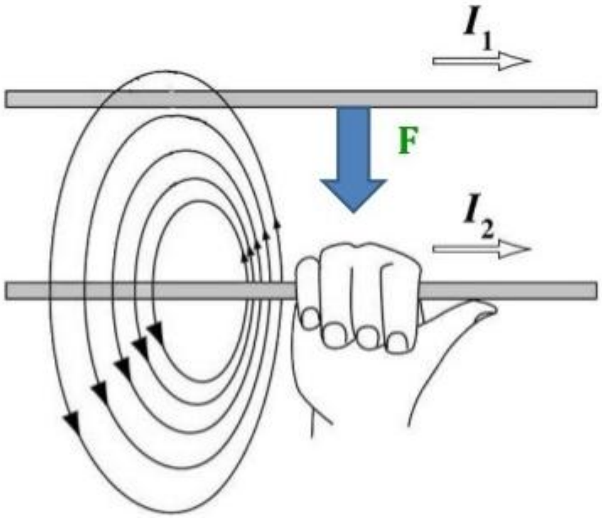
Space-charge

Transverse Interaction of Co-Moving Charges

Electric Repulsion:



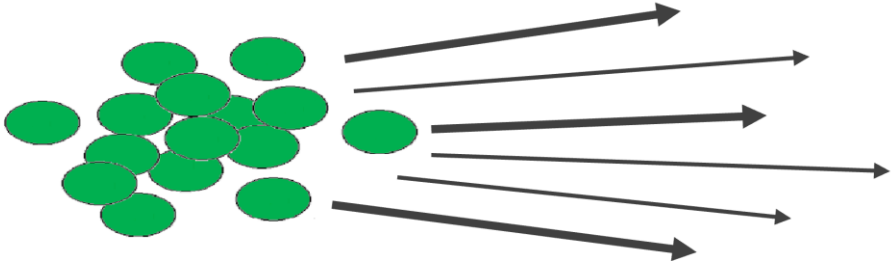
Magnetic Attraction:



$$\vec{E}_\perp = \gamma E'_\perp$$

$$\vec{B}_\perp c = \beta(\hat{z} \times \vec{E}_\perp)$$

Weakened Repulsion with Acceleration:



$$\vec{F}_\perp = q(E + v \times B)_\perp$$

$$\vec{F}_\perp = q(1 - \beta^2)E_\perp = \frac{q}{\gamma^2}E_\perp$$

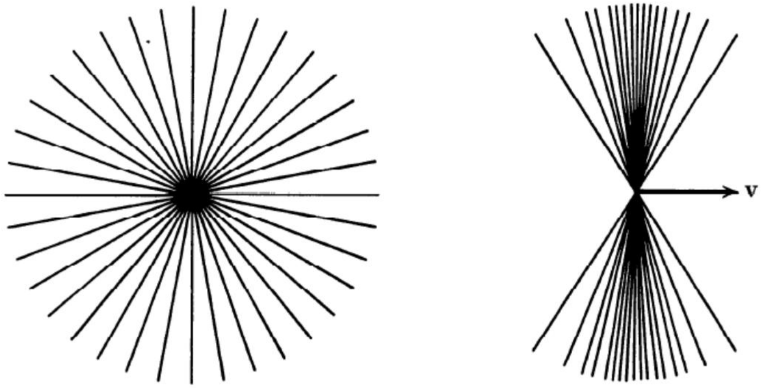
Space-charge Core vs Tail

Transverse space-charge forces much stronger than longitudinal space-charge.

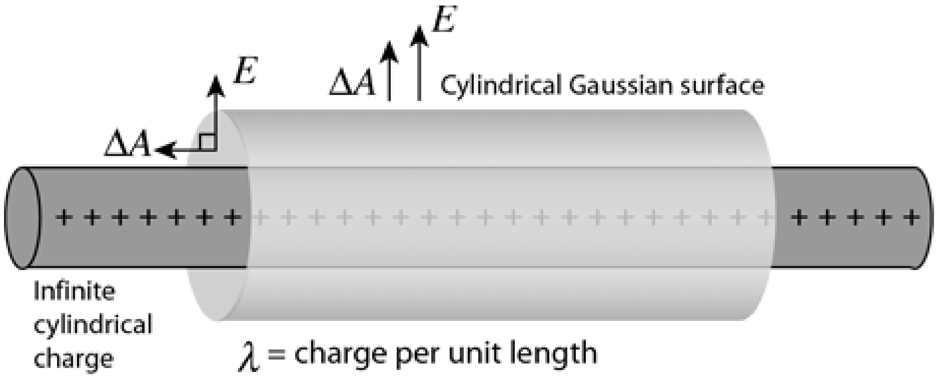
Transverse and Longitudinal charge distribution can be written as separable functions:

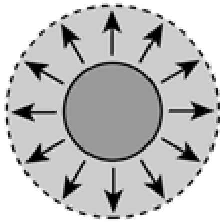
$$\rho(x,y,z) = \lambda(z)\rho_\perp(x,y)$$

Relativistic Distortion of EM Fields "Pancake-ification"



Gaussian cylinder for a line-charge:





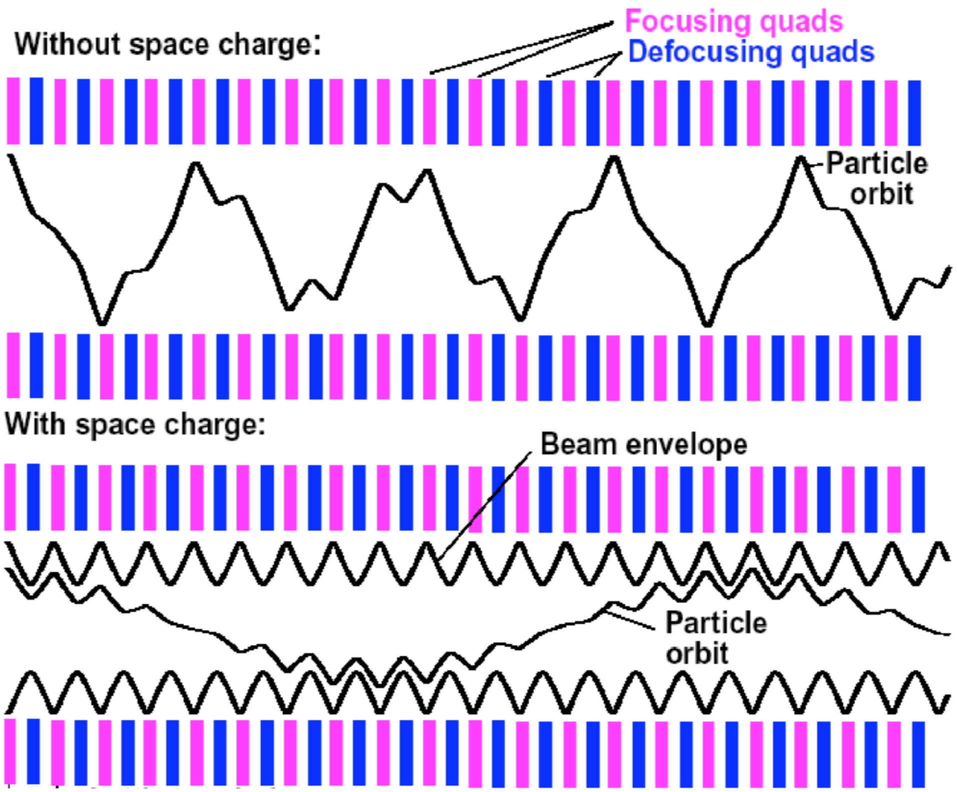
For $r < R$

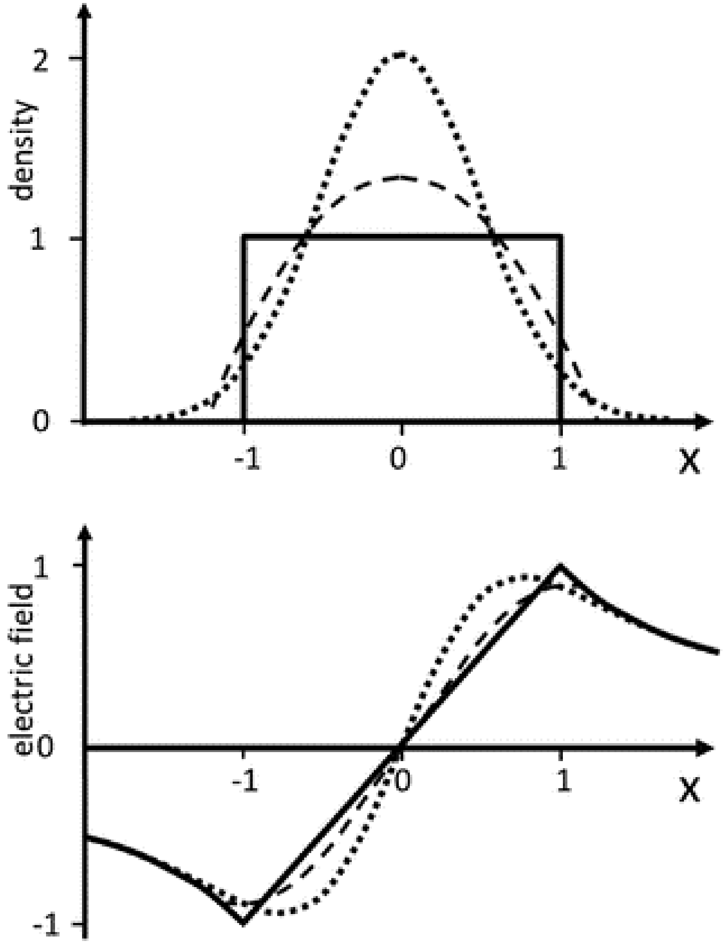
$$E = \frac{\lambda r}{2\pi\varepsilon_0 R^2}$$

For $r \geq R$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

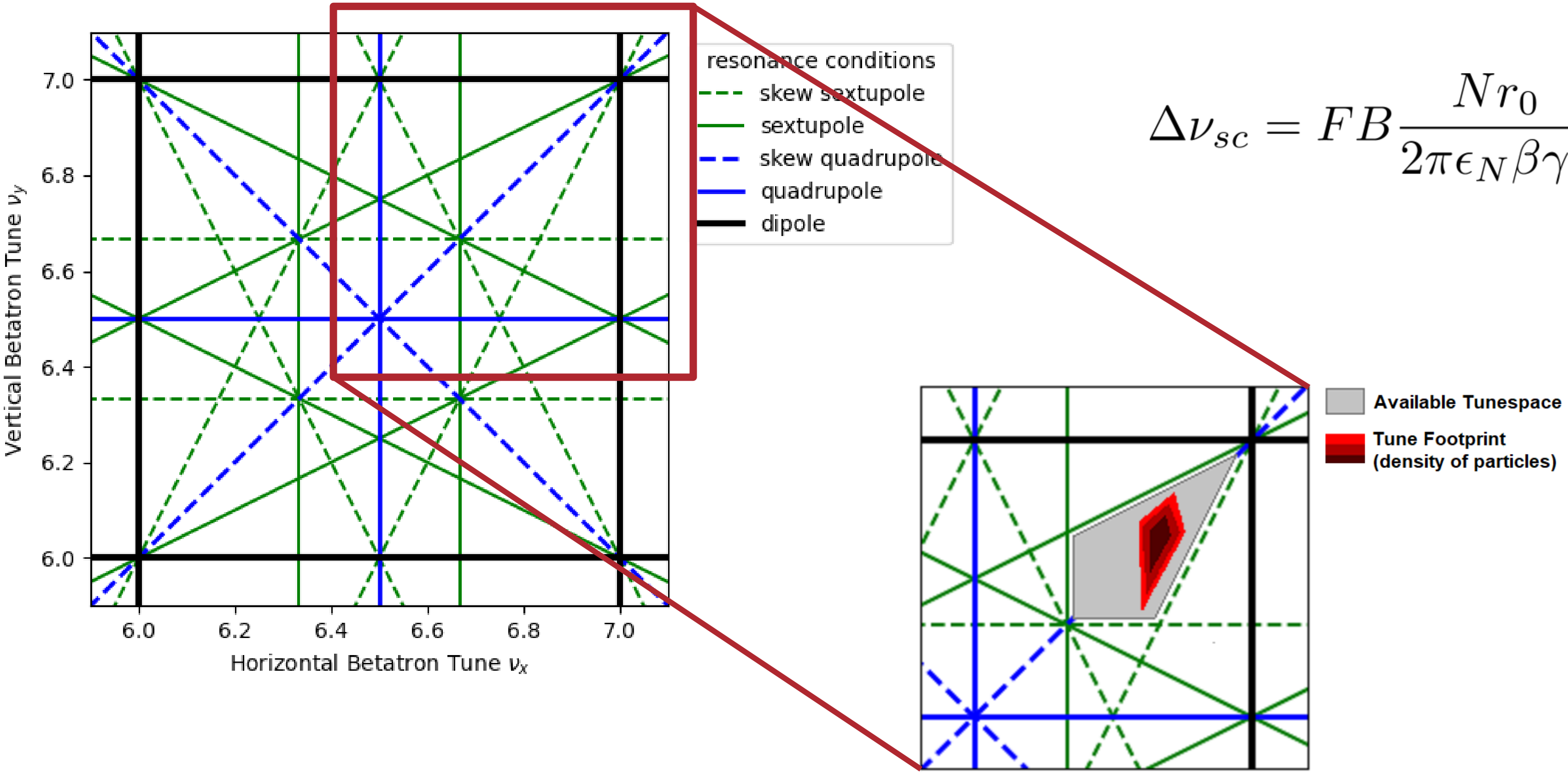
Space-charge Core vs Tail





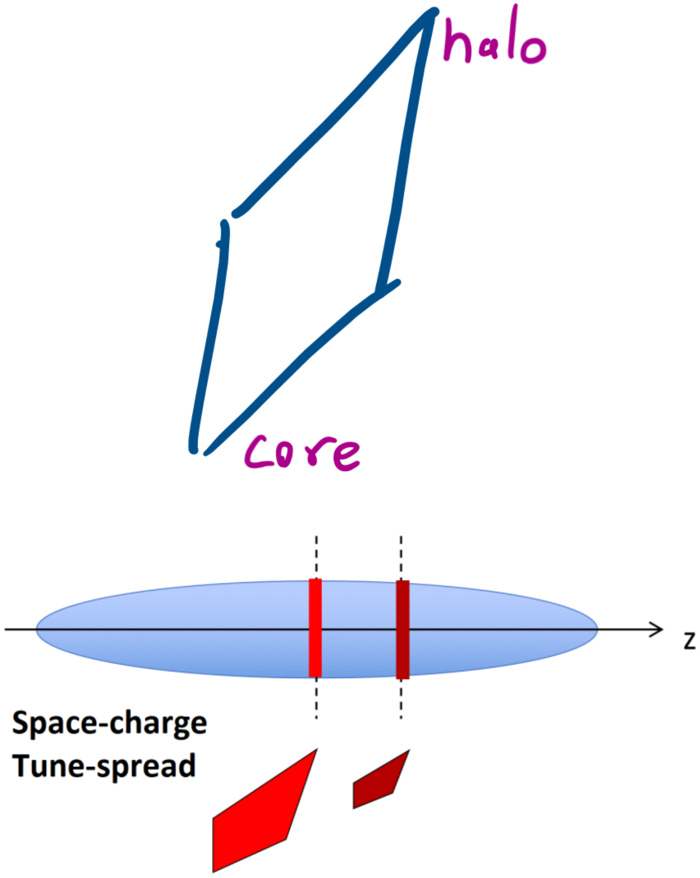
S. Lund

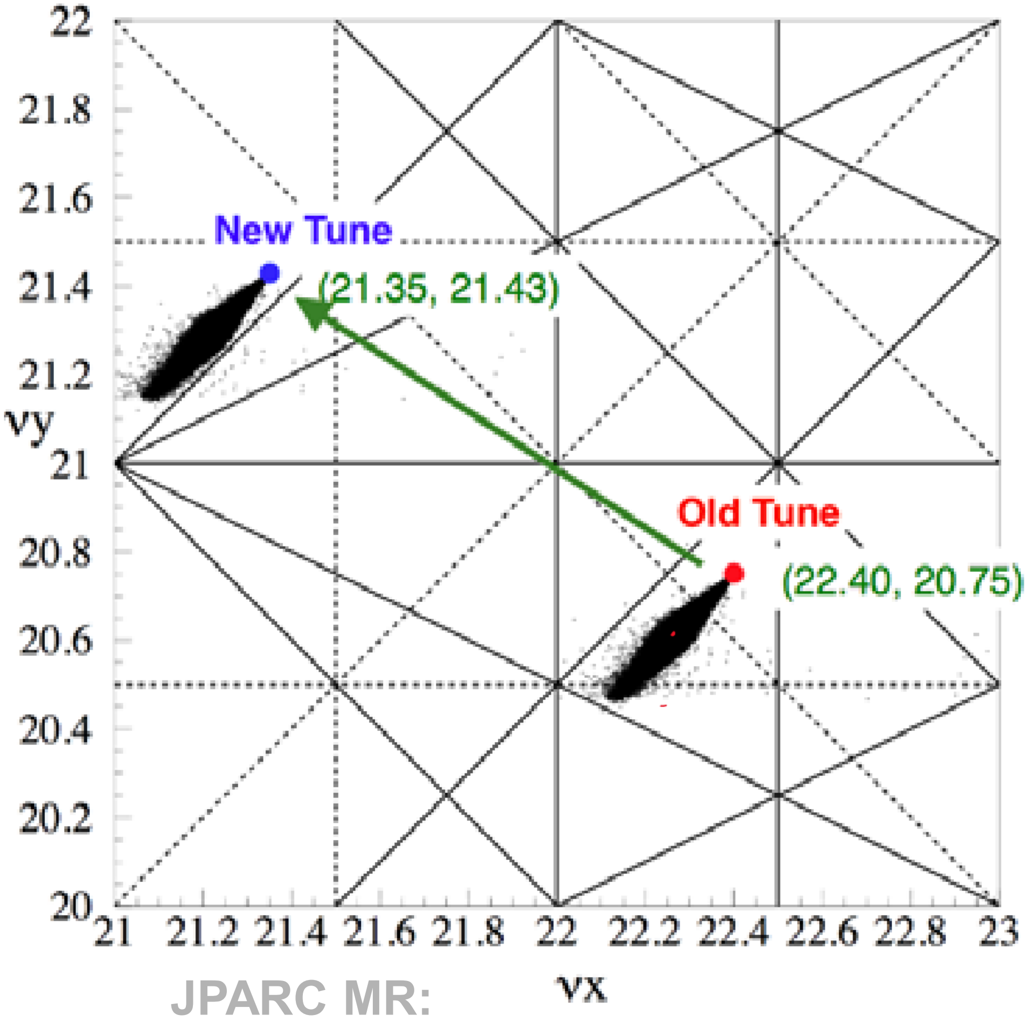
Tune Diagrams



$$\Delta\nu_{sc} = FB\frac{Nr_0}{2\pi\epsilon_N\beta\gamma^2}$$

Space-charge Tune-spread & Betatron Resonances





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PRSTAB 2017

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